Existence of Invariant Norms in p-adic Representations of Reductive Groups

Thesis submitted for the degree of Doctor of Philosophy

by

Eran Assaf

Submitted to the Senate of the Hebrew University of Jerusalem

November 2016

Existence of Invariant Norms in p-adic Representations of Reductive Groups

Thesis submitted for the degree of Doctor of Philosophy

by

Eran Assaf

Submitted to the Senate of the Hebrew University of Jerusalem

November 2016

This work was carried out under the supervision of
Prof. Ehud de Shalit

Acknowledgements

First and foremost, I would like to thank my advisor, Prof. Ehud de Shalit, for his guidance. Udi's intuition and ideas significantly aided my research, and helped me think differently about mathematics. Udi has also introduced me to the academic world in general, and important events in our field of research, in particular. His natural amity and optimistic views, combined with his willingness to improve processes in the university and in our society in general, has impressed and inspired me greatly.

Next, I would like to thank my thesis committee - Prof. David Kazhdan and Prof. Michael Schein - for their time and insights. In addition, I would like to thank David for introducing me to algebraic groups, and for his mathematical and meta-mathematical ideas and advice throughout my studies. Even though this work is very technical, David's views about simplicity in mathematics have helped me to see things more clearly.

I would also like to thank my fellow graduate students for their friendship, and many helpful talks and ideas. Special thanks are due to Uri Brezner and his profound understanding of algebraic geometry, to Adina Cohen for many fruitful discussions, to Amit Ophir and Oren Becker for our special seminar, to Cy Maor for tea and council, and to my officemates Yatir Halevi and Ohad Drucker.

Finally, I would like to thank my wife and family for their support all throughout my studies, which has enabled this research.

Abstract

This thesis lies in the framework of the p-adic Langlands programme, whose goal is to deepen our understanding about fundamental problems in number theory, such as the solution of Diophantine equations, the study of elliptic curves, and the study of Shimura varieties. The Langlands programme aims at creating a correspondence between certain representations of Galois groups and certain representations of the Adelic points of suitable reductive groups. An important and interesting particular case is class field theory, in which such a correspondence has been established, between characters of the Galois group and Dirichlet characters of the Ideles.

As a part of the Langlands programme, Robert Langlands has introduced the Local Langlands Conjectures, which describe a correspondence between complex representations of a reductive algebraic group G over a local field F and homomorphisms of the Galois group of F into the L-group of G. These conjectures have been proven for $G = GL(n)$ and for several other cases. This correspondence is preserved in l-adic representations, where l is not the characteristic of the residue field of F. However, when $l = p$ is the characteristic of the residue field of F, this correspondence no longer holds.

The purpose of the p-adic Langlands programme is to create a similar correspondence between some of the p-adic representations of the reductive group, and some of the p-adic representations of the Galois group. Such a correspondence has been shown for $G = GL(2)$ when the field is $\mathbb{Q}_p$. In this case, the representations of G which took part in the correspondence were Banach spaces with a G-invariant norm. In the cases where the representation of the Galois group is geometric, these spaces have algebraic vectors (that the G-action on them is locally polynomial) which form a locally algebraic representation of G with a G-invariant norm, dense in the original Banach space.

The relation between these spaces and the representations of the Galois group raised the possibility of generalizing the correspondence to other fields or other groups by looking at the locally algebraic representations of the reductive group G, finding G-invariant norms and completing with respect to these norms to obtain Banach Spaces which are the candidates to correspond to the appropriate representations of the Galois group.

Therefore, Breuil and Schneider conjectured a criterion for the existence of an invariant norm in a p-adic representation of the group $G = GL(n)$, and even generalized it to an arbitrary split reductive group.

In this work, we prove several special cases of this conjecture. The problems

considered here can be classified into two interconnected topics – the existence of invariant norms in locally algebraic representations of $GL(2)$ over a local field, and the existence of invariant norms in locally algebraic representations of $U(3)$ over a local field.

The methods employed in this thesis can also be classified into two essential types - one method is reduction of the problem to pure p-adic analysis, which we could only perform for $G = GL(2)$, while the other methods use the homology of the Bruhat-Tits tree of the reductive group G, and could be generalized to arbitrary unramified groups by employing similar methods on the Bruhat-Tits building, with increasing technical difficulty.

In Chapter 2, we prove the existence of an invariant norm in locally algebraic representations of $GL(2)$ over a local field, when the Breuil-Schneider criterion holds, for unramified representations of small weight, and for smooth tamely ramified representations. Even though both results have been known before, each result was proved in a different method, and this is the first method proving both.

In Chapter 3, we prove the existence of an invariant norm in locally algebraic representations of $GL(2)$ over a local field, when the Breuil-Schneider criterion holds, for some unramified representations of higher weights. Apart from the restriction on the weight, we have here a technical restriction on the representation, which we could not remove, but we estimate that it is purely technical.

In Chapter 4, we prove the existence of an invariant norm in locally algebraic representations of $U(3)$ over a local field, when the Breuil-Schneider criterion holds, for unramified representations of small weight, and for smooth tamely ramified representations. Each result is achieved using a different method.

None of the chapters have been published yet.

Letter of Contribution

This thesis contains three papers.

Two of the papers, included as chapters 3 and 4, are authored by Eran Assaf, and are the result of Eran Assaf's research during his PhD studies under the supervision of the advisor, Prof. Ehud de Shalit. Both papers are still unpublished.

The paper "Kirillov models and the Breuil-Schneider conjecture for $GL_2(F)$", included as chapter 2 in this thesis is authored by Eran Assaf, David Kazhdan and Ehud de Shalit, and remains unpublished. The following paragraphs clarify the different contributions of each author, as required by the Hebrew University in order to include the paper as a part of Eran Assaf's Ph.D. thesis.

David Kazhdan initiated the research project that eventually led to this paper. He suggested the idea of using Kirillov models in order to reduce the problem to p-adic analysis, but most mathematical work (including the precise definitions, claims and proofs) has been done by Ehud de Shalit and Eran Assaf, who are the two equal main contributors for this paper. This paper is a direct continuation and generalization of a previous paper authored by David Kazhdan and Ehud de Shalit, which only deals with the smooth case.

Roughly speaking, Eran Assaf's contribution to the paper lies mainly in the exceptional case and the simplification of the tamely ramified case, and Ehud de Shalit's contribution to the paper lies mainly in introducing the lattices, establishing the recursion relations, and proving the unramified case. In particular, Section 2, which describes the preliminaries on Fourier analysis, lattices and the Kirillov model, and Section 3, which describes the proof of the unramified case are due to Ehud de Shalit. Due to Eran Assaf are Section 4, which describes the proof in the tamely ramified case, and Section 5, which describes the proof in the exceptional case.

Prof. Ehud de Shalit

Eran Assaf

Contents

1 Introduction

The p-adic local Langlands programme

Let p be a prime number, and let n be a positive integer. Let F be a finite extension of $\mathbb{Q}_p$, and let W_F be the Weil group of F. Let l be a prime number, such that $l \neq p$.

Local class field theory gives an isomorphism between the abelianization of the Weil group, W_F^{ab} and the multiplicative group $F^\times$. Equivalently, we have a natural bijection between complex (resp. l-adic) continuous characters of W_F and complex (resp. l-adic) smooth characters of $F^\times$.

This theory has a non-abelian generalization, the local Langlands correspondence. The (classical) local Langlands correspondence for GL_n over p-adic fields, proved by Harris and Taylor [16], by Henniart [17], and later by Scholze [23] asserts the existence of a canonical bijection between the set of isomorphism classes of irreducible continuous complex (resp. continuous l-adic, see Vigneras [27]) representations of W_F of dimension n and the set of isomorphism classes of complex (resp. l-adic) smooth irreducible supercuspidal representations of $GL_n(F)$.

This correspondence is moreover compatible with reduction modulo l ([25]) and with cohomology ([16]).

The original aim of the local p-adic Langlands programme is to look for a possible p-adic analogue of the classical and l-adic correspondence, stated in the previous subsection.

Note that when considering continuous p-adic representations, Grothendieck's l-adic monodromy theorem no longer holds, and we might have wild ramification that cannot be solved by passage to a finite extension of F. This shows that we have a richer category of representations of W_F in the p-adic case, while the category of smooth representations of $GL_n(F)$ does not see the field of coefficients, hence remains the same. This suggests that one needs to enlarge the category considered on the reductive side, and that one indeed needs a p-adic correspondence, which essentially differs from the l-adic correspondence.

The local p-adic correspondence for $GL_2(\mathbb{Q}_p)$ was fully developed, essentially by Berger, Breuil and Colmez in [2], [8] and completed by Colmez, Dospinescu and Paskunas in [9], using the theory of (φ, Γ)-modules.

The p-adic local Langlands correspondence associates to certain 2-dimensional continuous representations of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, certain Banach spaces equipped with a unitary continuous action of $GL_2(\mathbb{Q}_p)$. It has three important compatibility properties:

- compatibility with reduction modulo p [1]
- compatibility with classical local Langlands correspondence [8, 12]
- local-global compatibility with completed étale cohomology [12]

These properties already have remarkable global applications. For example, Kisin shows in [20] that the compatibility with the classic local Langlands correspondence, under some weak technical assumptions, implies the Breuil-Mézard conjecture on modular multiplicities [5]. Combined with the proof of Serre's modularity conjecture by Khare-Winterberger-Kisin, Emerton's local-global compatibility then allows one to prove many cases of the Fontaine-Mazur conjecture, which characterizes the representations of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from classical modular forms.

If F is a finite extension of $\mathbb{Q}_p$, it is natural to ask how to associate p-adic representations of $GL_2(F)$ to 2-dimensional p-adic representations of $Gal(\overline{F}/F)$. This problem turns out to be far more delicate when $F \neq \mathbb{Q}_p$ and even the theory modulo p for $GL_2(F)$ is very much involved. Furthermore, as Colmez's technique in [8] specifically associates to a (φ, Γ)-module a 2-dimensional representation of the Galois group, it is not clear yet how to approach the case of $GL_n(F)$ for $n > 2$, even when $F = \mathbb{Q}_p$, or more generally, the F-points of an arbitrary reductive group G.

In order to obtain a first approximation to what we might expect in such cases, we will first introduce some notions regarding Banach space representations of reductive groups, and then recall some of the constructions arising when $n = 2$ and $F = \mathbb{Q}_p$.

Representations of Reductive Groups

Let C be a field of characteristic 0, used as the field of coefficients.

Let $\mathbf{G}$ be a connected reductive group defined over F, a finite extension of $\mathbb{Q}_p$, and set $G = \mathbf{G}(F)$.

Definition 1. *Smooth Representation*

A representation $\pi : G \to GL(V)$ on a C-vector space V is called smooth *if the stabilizer of each vector $v \in V$ is open in G.*

Definition 2. *Smooth Admissible Representation*

A smooth representation $\pi : G \to GL(V)$ on a C-vector space V is called admissible *if for every open subgroup $H \subseteq G$ the space V^H of H-invariants in V is finite-dimensional.*

The following theorem is highly important in the development of the theory of smooth representations, and was proved by Jacquet in [19]

Theorem 3. *Let (π, V) be an irreducible smooth representation of G. Then (π, V) is admissible.*

We now focus our attention on the case where C is p-adic.

Assume from now on that C is a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_C$ its ring of integers. Let $\varpi \in \mathcal{O}_C$ be a uniformizer, and $\kappa_C = \mathcal{O}_C/\varpi\mathcal{O}_C$ the residue field, of cardinality q. Let $|\cdot|$ denote the absolute value on C, normalized so that $|\varpi| = q^{-1}$.

As we mentioned before, in this case, it is reasonable to introduce larger classes of representations, since enriching the category of representations of the reductive groups is necessary for extending the local Langlands correspondence to p-adic Galois representations. First, as in the l-adic case (see Vigneras, [27]), we introduce the notion of a unitary Banach space representation.

Definition 4. *Unitary Banach Space Representation*

Let $(V, \|\cdot\|)$ be a Banach C-vector space. A Unitary C-Banach space representation *of G (on V) is a G-action by continuous linear automorphisms such that the map $G \times V \to V$ giving the action is continuous, and given by isometries, i.e. for all $g \in G$ and all $v \in V$*

$$\|gv\| = \|v\|.$$

In order to motivate what follows, we would like to mention two pathologies of Banach space representations:

- There exist non-isomorphic topologically irreducible Banach space representations V and W of G for which nevertheless there is a nonzero G-equivariant continuous linear map $V \to W$ (similarly to the case of real Lie groups).
- Even such a simple commutative group such as $G = \mathbb{Z}_p$ has infinite dimensional topologically irreducible Banach space representations (see [11]).

It is clear that in order to avoid such pathologies we have to impose an additional finiteness condition on our Banach space representations. This condition will be called admissiblity, and in a series of papers, Schneider and Teitelbaum found out what seems to be the right notion (see [21]).

Definition 5. *Admissibility*

Let G be a finite dimensional p-adic Lie group. A C-Banach space representation V of G is called admissible *if there is a G-invariant bounded open lattice $L \subseteq V$ such that for any open normal subgroup $H \subseteq G$ the κ_C-vector space $(L/\varpi L)^H$ of H-invariant elements in $L/\varpi L$ is finite dimensional.*

We point out that this condition implies that V is admissible if and only if there is a G-invariant bounded open lattice $L \subseteq V$ such that $L/\varpi L$ is an admissible smooth representation of G over the residue field κ_C.

The following proposition, which is proved by Schneider and Teitelbaum in [21, Lemma 3.4] and the discussion following it, shows that it is enough to check the condition for a single open normal subgroup $H \subseteq G$ which is pro-p. Moreover, Schneider and Teitelbaum show that if one considers the invariants by such an open normal pro-p subgroup, one can take any open G-invariant lattice.

Proposition 6. *Let V be a C-Banach space representation of G. Let $H \subseteq G$ be an open normal pro-p subgroup. Assume that there is a G-invariant bounded open lattice $L \subseteq V$ such that the κ_C-vector space $(L/\varpi L)^H$ of H-invariant elements in $L/\varpi L$ is finite dimensional. Then V is admissible.*

Conversely, if V is admissible, then for any G-invariant bounded open lattice $L \subseteq V$, the κ_C-vector space $(L/\varpi L)^H$ of H-invariant elements in $L/\varpi L$ is finite dimensional.

The p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ (see [2], [8] and [9]) gives a bijection between isomorphism classes of 2-dimensional absolutely irreducible continuous representations of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over C and isomorphism classes of absolutely irreducible, non-ordinary, admissible unitary Banach representations of $GL_2(\mathbb{Q}_p)$ over C.

Restricting the bijection to the potentially semi-stable Galois representations, with distinct Hodge-Tate weights, one obtains an important subcategory on the reductive side, consisting of completions of irreducible locally algebraic representations.

For this reason, we proceed to define locally algebraic representations. But first, let us define an algebraic representation.

Let

$$\widetilde{\mathbf{G}} := \left(Res_{F/\mathbb{Q}_p}\mathbf{G}\right)_C$$

be the reductive group over C obtained by base extension from the Weil restriction from F to $\mathbb{Q}_p$ of $\mathbf{G}$. Write

$$\widetilde{G} := \widetilde{\mathbf{G}}(C) = \mathbf{G}(F \otimes_{\mathbb{Q}_p} C).$$

The ring homomorphism $F \to F \otimes_{\mathbb{Q}_p} C$ which sends a to $a \otimes 1$ induces an embedding of groups $G \hookrightarrow \widetilde{G}$.

Definition 7. *Let (τ, V) be a representation of G on a C-vector space. Then τ is* algebraic *if there is a rational representation $\widetilde{\tau}$ of $\widetilde{\mathbf{G}}$ on V such that τ is the pullback of $\widetilde{\tau}$ via $G \hookrightarrow \widetilde{G}$.*

We now turn to the definition of a locally algebraic representation.

Definition 8. *Let (π, V) be a representation of G over C. We say that $v \in V$ is* locally algebraic *if there exists a compact open subgroup $K_v \subseteq G$ and a finite dimensional subspace $U \subseteq V$ containing the vector v such that K_v leaves U invariant and operates on U via restriction to K_v of a finite dimensional algebraic representation of G. We denote by $V^{alg} \subseteq V$ the subspace of locally algebraic vectors. If $V = V^{alg}$, we say that V is* locally algebraic.

An important result, proved by Prasad in [22, Appendix], gives an explicit description of the irreducible locally algebraic representations of G.

Theorem 9. *Let π be an irreducible locally algebraic representation of G. Then there is an irreducible algebraic representation τ of G, and an irreducible smooth representation σ of G, such that $\pi = \tau \otimes \sigma$. Conversely, if τ and σ are as above, then π is an irreducible locally algebraic representation of G.*

This will allow us to restrict our attention mostly to representation of the form $\pi = \tau \otimes \sigma$.

The Breuil-Schneider Conjecture

We would like to formulate a p-adic correspondence for arbitrary n and F. In order to have an idea as to how such a correspondence should be obtained, let us take another look at the construction of the correspondence for $GL_2(\mathbb{Q}_p)$.

In general, for potentially semi-stable n-dimensional continuous representations of the absolute Galois group $Gal(\overline{F}/F)$, it is possible to attach a smooth representation $\sigma = \sigma(\rho)$ of

$GL_n(F)$, as in the classical case. However, $\rho \rightsquigarrow \sigma(\rho)$ is no longer reversible. Nevertheless, as the coefficient field is now an extension of $\mathbb{Q}_p$, if we assume that ρ has distinct Hodge-Tate weights, we may also construct an irreducible algebraic representation $\tau = \tau(\rho)$ of $GL_n(F)$ from the Hodge-Tate weights of ρ. We will specify the explicit construction when formulating the conjecture subsequently.

Still, one cannot reconstruct ρ from $\sigma(\rho)$ and $\tau(\rho)$. The problem is that the semi-stable representations ρ are classified by their filtered (φ, N)-modules, and not only by their (φ, N)-modules and the Hodge-Tate weights. The Hodge filtration is lost when constructing the representations $\tau(\rho)$ and $\sigma(\rho)$. Note that as the coefficient field is p-adic, these two representations live in the same universe, and it makes sense to consider the representation $\pi = \sigma \otimes \tau$. These representations are no longer smooth, neither are they algebraic, but they are locally algebraic.

The p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ takes any continuous representation $\rho : Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(\overline{\mathbb{Q}}_p)$ and attaches to it a unitary Banach space $\Pi(\rho)$ with an admissible unitary $GL_2(\mathbb{Q}_p)$-action. This map $\rho \rightsquigarrow \Pi(\rho)$ is reversible, and compatible with classical local Langlands in the following sense: When ρ is potentially semistable with distinct Hodge-Tate weights,

$$\Pi(\rho)^{alg} = \tau(\rho) \otimes_{\overline{\mathbb{Q}}_p} \sigma(\rho)$$

Furthermore, $\Pi(\rho)^{alg} = 0$ otherwise (see [8, Theorem VI.6.13]).

When ρ is irreducible, $\Pi(\rho)$ is known to be the completion of $\tau(\rho) \otimes_{\overline{\mathbb{Q}}_p} \sigma(\rho)$ relative to a suitable $GL_2(\mathbb{Q}_p)$-invariant norm $\|\cdot\|$ which corresponds to the lost Hodge filtration.

As for a general field F, if $\rho : Gal(\overline{F}/F) \to GL_n(\overline{\mathbb{Q}}_p)$ is potentially semi-stable , one can define the representation $\pi = BS(\rho) := \tau(\rho) \otimes_{\overline{\mathbb{Q}}_p} \sigma(\rho)$. This representation is a dense subrepresentation of a unitary $GL_n(F)$ Banach representation $\Pi(\rho)$ if and only if it admits a $GL_n(F)$-invariant norm.

This suggests, at least as a first approximation, that the existence of such an invariant norm will be equivalent to having a representation corresponding to a potentially semi-stable representation.

We now formulate the conjecture more precisely.

Let $G = GL_n(F)$, $n \geq 2$. Let F' be a finite Galois extension of F, and F_0' its maximal unramified subfield. Assume $[F_0' : \mathbb{Q}_p] = |Hom_{\mathbb{Q}_p}(F_0', C)|$ and let $p^{f'}$ be the cardinality of the residue field of F_0' and φ_0' the Frobenius on F_0' (raising to the p-th power each component of the Witt vectors).

Let $Mod_{F'/F}$ be the category of discrete finite dimensional $(\varphi, N, Gal(F'/F))$-modules. Let $WD_{F'/F}$ be the category of finite dimensional Weil-Deligne representations over C which are unramified when restricted to $W(\overline{F}/F')$. Fontaine, in [14], constructs a functor $WD : Mod_{F'/F} \to WD_{F'/F}$ which induces an equivalence of categories.

Now, if (ρ, N, V) is an object of $WD_{F'/F}$ such that ρ is semisimple, we have by the classical local Langlands correspondence, a smooth irreducible representation of G over $\overline{\mathbb{Q}}_p$, σ^{unit} corresponding to (ρ, N, V), normalized so that the central character of σ^{unit} is $\det(r, N, V) \circ Art_F^{-1}$, with Art_F being Artin's reciprocity map from local class field theory, sending uniformizers to geometric Frobenii. Note that σ^{unit} depends on a choice of $q^{1/2}$ in $\overline{\mathbb{Q}}_p$.

Breuil and Schneider in [6] construct a modification, σ of σ^{unit}, which does not depend on the choice of $q^{1/2}$, and is a smooth representation of G over C. If (ρ, N, V) is an object of $WD_{F'/F}$ we denote by $(\rho, N, V)^{ss}$ its ρ-semisimplification.

Conjecture 10. *The Breuil-Schneider Conjecture*

Fix an object (ρ, N, V) of $WD_{F'/F}$ such that ρ is semisimple. For each $\iota : F \hookrightarrow C$, fix a list of n integers $i_{1,\iota} < i_{2,\iota} < \ldots < i_{n,\iota}$. Let σ be the smooth representation of G over C described above. Let $a_{j,\iota} = -i_{n+1-j,\iota} - (j-1)$ and denote by τ the unique $\mathbb{Q}_p$-rational representation of G over C such that $\tilde{\tau} = \otimes_\iota \tau_\iota$ with τ_ι of highest weight $(a_{1,\iota}, \ldots, a_{n,\iota})$.

The following conditions are equivalent:

1. *There is an invariant norm on $\tau \otimes_C \sigma$.*
2. *There is a $(\varphi, N, Gal(F'/F))$-module D such that*

$$WD(D)^{ss} = (\rho, N, V)$$

and an admissible filtration $(Fil^i D_{F',\iota})_{i,\iota}$ preserved by $Gal(F'/F)$ on $D_{F'} := F' \otimes_{F_0} D$ such that $\forall \iota : F \hookrightarrow C$

$$Fil^i D_{F',\iota} / Fil^{i+1} D_{F',\iota} \neq 0 \iff i \in \{i_{1,\iota}, \ldots, i_{n,\iota}\}.$$

This asserts that the existence of an invariant norm on $\tau \otimes_C \sigma$ is equivalent to the existence of a (weakly) admissible filtered $(\varphi, N, Gal(F'/F))$-module, the semisimplification of its image being (ρ, N, V), namely (ρ, N, V) is becoming semi-stable over F'.

By the equivalence of categories stated by Fontaine, and proved by Colmez-Fontaine, the conjecture predicts that the existence of an invariant norm on $\tau \otimes_C \sigma$ is equivalent to the existence of a potentially semi-stable representation V of $\mathcal{G}_F$ of dimension n over C, such that its Hodge-Tate weights are the $-i_{j,\iota}$ and such that the ρ-semisimplification of its associated Weil-Deligne representation has σ as a Langlands parameter (modified as above).

The "if" part is completely known for $GL_n(F)$ ([18]), and is due to Y. Hu. The "only if" part remains open, even for $n = 2$.

Next, we introduce the notion of an integral structure, whose existence in a representation is equivalent to the existence of a G-invariant norm.

Definition 11. *Integral Structure*

Let V be a representation of G on a C-vector space. An integral structure *in V is an $\mathcal{O}_C[G]$-submodule which spans V over C and contains no C-line.*

An integral structure is also referred to in the literature as a *separated lattice*.

Note that asking for a norm in a representation V over C, a finite extension of F, amounts to asking for an integral structure, that is a sub-$\mathcal{O}_C[GL_n(F)]$-module generating V over C which contains no C-line: Given a norm $\|\cdot\|$, the unit ball is an integral structure. Conversely, given an integral structure Λ, its gauge $\|x\| = q_C^{-v_\Lambda(x)}$, where $v_\Lambda(x) = \sup\{v \mid x \in \varpi^v \Lambda\}$ is a norm. Thus we are looking for integral structures in locally algebraic representations of $GL_n(F)$.

The equivalence of norms gives rise to an equivalence relation on lattices, called commensurability. Explicitly, two integral structures in a representation are *commensurable* if each of them is contained in a scalar multiple of the other. Note that any two finitely generated integral structures are commensurable, hence of minimal nonzero commensurability class.

Already in [6], the authors discuss a generalization to the case of an arbitrary split reductive group G. In fact, they construct a Banach space for any pair (ξ, ζ) where ξ is a dominant weight of the split torus T and $\zeta \in T'$, the torus of the Langlands dual group. Given this pair, they construct a family of p-adic Galois representations with values in the Langlands dual group G'. Conjecture 10 is then equivalent to asserting that this Banach space is nonzero for $G = GL_n(F)$. We note that this conjecture can be formulated purely in terms of the reductive group G, ignoring the original Weil-Deligne representation. In order to do so, we recall Emerton's treatment of these ideas ([13]).

Emerton's condition and arbitrary reductive groups

In [22], Prasad shows that any irreducible locally algebraic representation of a p-adic reductive group, G, is of the form $\sigma \otimes \tau$ with σ smooth and τ algebraic. Moreover, $V = \sigma \otimes \tau$ is irreducible if and only if both σ and τ are irreducible. If V admits a G-invariant norm, the central character must be unitary. Let P be a parabolic subgroup of G, with unipotent radical N and Levi quotient M. Let N_0 be some compact open subgroup of N. Let δ denote the modulus character of P (which is trivial on N, and so induces a character of $M = P/N$, which we also denote by δ; concretely, $\delta(m) = [mN_0m^{-1} : N_0]$). Let $J_P(V)$ denote Emerton's Jacquet module (with respect to P) of V, i.e. if $V = \sigma \otimes \tau$, then

$$J_P(V) = \tau^N \otimes_C (res_P^G \sigma)_N \delta^{1/2}$$

Let Z_M be the center of M. Write $Z_M^+ := \{z \in Z_M \mid zN_0z^{-1} \subset N_0\}$.

Lemma 12. *(Emerton)*

Let χ be a locally algebraic C-valued character of Z_M. If the χ-eigenspace of $J_P(V)$ is nonzero, and V admits a G-invariant norm, then

$$\left|\chi(z)\delta^{-1}(z)\right| \leq 1 \quad \forall z \in Z_M^+$$

In [18], Hu shows that this is equivalent, in the case of $GL_n(F)$, to the requirement that V arises from a potentially semi-stable Galois representation. Thus, it makes sense to reformulate the conjecture for arbitrary reductive groups.

Conjecture 13. *Assume that for any locally algebraic character $\chi : Z_M \to C^\times$ with $J_P(V)_\chi \neq 0$,*

$$\left|\chi(z)\delta^{-1}(z)\right| \leq 1 \quad \forall z \in Z_M^+$$

and that the central character of V is unitary. Then V admits a G-invariant norm.

Progress on the Breuil-Schneider conjecture

- Note that the central character of $BS(\rho)$ always attains values in $\mathcal{O}_C^{\times}$. Sorensen ([24]) has proved for any connected reductive group G defined over $\mathbb{Q}_p$, that if τ is an irreducible algebraic representation of $G(\mathbb{Q}_p)$, and σ is an essentially discrete series representation of $G(\mathbb{Q}_p)$, both defined over C, then $\tau \otimes_C \sigma$ admits a $G(\mathbb{Q}_p)$-invariant norm if and only if its central character is unitary.

- On 2013 there has been spectacular progress on the BS conjecture in the principal series case, which is the most difficult, by joint work of Caraiani, Emerton, Gee, Geraghty, Paskunas and Shin ([7]). Using global methods, they construct a candidate Π, which could depend on some global data in addition to ρ, for a p-adic local Langlands correspondence for $GL_n(F)$ and are able to say enough about it to prove new cases of the conjecture. Their conclusion is even somewhat stronger than the existence of a norm on $BS(\rho)$, in that it asserts admissibility.

Both works employ global methods, and as this is a question of local nature, we believe that there must be some local method to recover these results. There has also been some progress employing local methods, which yields results also for finite extensions of $\mathbb{Q}_p$, namely:

- For $GL_2(F)$, Vigneras ([26]) constructed an integral structure in tamely ramified smooth principal series representations, satisfying the assumption that they arise from p-adic potentially semistable Galois representations.

- For $GL_2(\mathbb{Q}_p)$, Breuil ([4]) used compact induction together with the action of the spherical Hecke algebra to produce a separated lattice in $BS(\rho)$ where $\pi = BS(\rho)$ is an unramified locally algebraic principal series representation, under some technical p-smallness condition on the weight. This was later generalized to $GL_2(F)$ by de Ieso ([10]).

- For general split reductive groups, Große-Klönne ([15]) looked at the universal module for the spherical Hecke algebra, and was able to show some cases of the conjecture for unramified principal series, again under some p-smallness condition on the Coxeter number (when $F = \mathbb{Q}_p$) plus other technical assumptions.

New cases of the Breuil-Schneider conjecture

Let p be a prime number. Let F be a finite extension of $\mathbb{Q}_p$. Fix a uniformizer ϖ of F, and let q be the cardinality of its residue field $\kappa_F = \mathcal{O}_F/\varpi\mathcal{O}_F$. Let $v_F : F^\times \to \mathbb{Z}$ be the valuation on F normalized so that $v_F(\varpi) = 1$. Let C be a finite extension of F. Here and in what follows, we fix one embedding $F \hookrightarrow C$ and consider the special case of Conjecture 10 where the algebraic representation $\tilde{\tau}$ is trivial for the other embeddings.

Let $\mathbf{G}$ be a reductive group over F and let $G = \mathbf{G}(F)$ be the group of its F-points.

In this work, we consider the principal series representations of the reductive groups G, and prove the existence of invariant norms in some of these representations. The case of locally algebraic principal series representations seems to be the most difficult when considering the existence of G-invariant norms, or equivalently an integral structure.

Quite generally, if $\mathbf{G}$ is an arbitrary reductive group, and $\pi = \tau \otimes \sigma$ is an irreducible locally algebraic representation of $G = \mathbf{G}(F)$, the simpler σ is algebraically, the harder the question of existence of G-invariant norms in π becomes. An obvious necessary condition is for the central character of π to be unitary, i.e. attain values in $\mathcal{O}_C^\times$. Assume therefore this is the case. If σ is supercuspidal (its matrix coefficients are compactly supported modulo the center), the existence of a G-invariant norm is obvious. As mentioned above, using global methods and the trace formula, existence of a G-invariant norm can also be proved when σ is essentially discrete series (its matrix coefficients are square integrable modulo the center) [24]. In these cases, no further restrictions are imposed on π. At the other extreme stand principal series representations, where one should impose severe restrictions on π, and the problem becomes very difficult. We therefore focus our attention on these representations.

In this thesis we show the existence of a G-invariant norm by showing the existence of an integral structure, hence the completion with respect to the resulting norm is nonzero. However, we do not show admissibility of the resulting completion. In some cases, we consider a finitely generated integral structure, hence the resulting completion is necessarily the universal completion, which is known to be non-admissible, for example, when $F \neq \mathbb{Q}_p$ (see [3]).

We consider two specific groups - the group $GL_2(F)$ is considered in chapters 2 and 3, while in chapter 4 we consider the unitary group $U_3(F)$.

Studying the Kirillov Model for $GL_2(F)$

In chapter 2, we provide a proof of two results, which were until now proved in two different methods, using a unified and new framework. We prove the Breuil-Schneider conjecture for both smooth tamely ramified principal series representations and unramified locally algebraic principal series representations of small weight. This is achieved by looking at the Kirillov model of the representation, and transforming the question on integral structures to a problem in p-adic analysis.

In order to state the result more precisely, let χ_1, χ_2 be smooth characters of $F^\times$ attaining values in $C^\times$, let B be the Borel subgroup of upper triangular matrices in $G = GL_2(F)$ and let

$$\sigma = Ind_B^G(\chi_1, \chi_2)$$

be the smooth (not normalized!) principal series representation induced from the character $\chi_1 \otimes \chi_2$, viewed as a character on the torus of diagonal matrices, and inflated to B. Explicitly, we require $f(bg) = (\chi_1 \otimes \chi_2)(b) \cdot f(g) \forall b \in B, g \in G$. Assume that χ_1, χ_2 are chosen such that σ is irreducible. Fix integers m and $n \geq 0$, and let

$$\tau = \det(\cdot)^m \otimes Sym^n$$

where Sym^n denotes the n-th symmetric power of the standard representation of $\mathbf{G}$. We use the space of univariate polynomials of degree at most n, $C[u]^{\leq n}$, as a model for the representation Sym^n.

The Breuil-Schneider conjecture for $\pi = \tau \otimes \sigma$ asserts that π admits an integral structure if and only if the following two conditions are satisfied:

(i) $\left|\chi_1(\varpi)\chi_2(\varpi)\varpi^{2m+n}\right| = 1$

(ii) $1 \leq |\chi_1(\varpi)\varpi^m| \leq \left|q^{-1}\varpi^{-n}\right|$

It is known that these two conditions are necessary. In Chapter 2, we prove the following theorem.

Theorem 14. *Assume that (i) and (ii) are satisfied. Assume, in addition, that either*

(1) *χ_1 and χ_2 are unramified and $n < q$, or*

(2) *χ_1 and χ_2 are tamely ramified and $n = 0$.*

Then π admits an integral structure.

Each of the cases (1) and (2) was already known ([26], [4], [10]), but treated separately. Here, using the study of the Kirillov model of the representation π, one can transform the problem to a problem in p-adic functional analysis, and use it to prove both cases. The restrictions on ramification level and algebraic weight are also needed in the aforementioned works. In fact, this is true even though Breuil, de Ieso and Vigneras all use the method of compact induction, working on the Bruhat-Tits tree of $G = GL_2(F)$, while our approach takes place in a certain dual space.

The Kirillov model $\mathcal{K}$ of π is the space of functions on $F^\times$

$$\mathcal{K} = C_c^\infty(F, \tau)\chi_1 + C_c^\infty(F, \tau)\omega\chi_2$$

where ω is the norm character, namely the unramified character satisfying $\omega(\varpi) = q^{-1}$, and $C_c^\infty(F, \tau)$ is the space of smooth compactly supported functions on F with values in τ. For the action of G on this model, see Chapter 2, Section 1.3.

Fourier analysis implies that an arbitrary function $\phi \in \mathcal{K}$ may be expanded annulus-by-annulus as

$$\phi = \sum_{l=l_0}^{\infty} \sum_{\beta \in F/\mathcal{O}_F} C_l(\beta)\phi_{l,\beta}$$

for some $l_0 \in \mathbb{Z}$, some predetermined functions $\phi_{l,\beta} \in C_c^\infty(F, C)$, which are supported on $\varpi^l \mathcal{O}_F^\times$ and the coefficients $C_l(\beta) \in \tau$ satisfy certain explicit recursion relations.

The theorem is based on the following observation. Let the assumptions be as in the theorem. If V_0 is an $\mathcal{O}_C[G]$-module spanned by a nonzero vector in the Kirillov model $\mathcal{K}$, one can find a family of $\mathcal{O}_C$-lattices $(M_0(\beta))_{\beta \in F/\mathcal{O}_F}$ in τ, such that if $\phi \in V_0$ vanishes outside of $\mathcal{O}_F$, it has an expansion as above with $C_0(\beta) \in M_0(\beta)$ for all β.

We note that while this claim is sufficient to prove the theorem, it is not clear that they are equivalent.

The first step in the proof of this theorem is standard, showing commensurability between V_0 and a certain $\mathcal{O}_C[B]$-module of finite type Λ, which is spanned by an explicit set of nice functions $\mathcal{E}$. If one considers $\phi \in \Lambda$ and expresses it as a linear combination of the functions in $\mathcal{E}$, and expand it annulus-by-annulus, the $C_l(\beta)$ satisfy certain recursive relations. If one further assumes that ϕ vanishes away from $\mathcal{O}_F$, we must have some cancellation. By increasing induction, one shows that for $l \leq 0$, the $C_l(\beta)$ belong to a certain $\mathcal{O}_C$-lattice $M_l(\beta)$ in τ, which depends on l and β, but not on ϕ.

The main phenomenon which assists us in establishing the theorem is that there are two distinct families of lattices, which coincide when $n < q$. In fact, for any $\beta \in F/\mathcal{O}_F$ and any integer l, one may define the disc

$$D_l(\beta) = \left\{\alpha \in F \mid \left|\alpha - \varpi^{-l}\beta\right| \leq \left|\varpi^{-l}\right|\right\}$$

and consider

$$N_l(\beta) = \left\{P \in C[u]^{\leq n} \mid |P(\alpha)| \leq |\varpi|^{-nl} \quad \forall \alpha \in D_l(\beta)\right\}$$

This family of lattices has the nice property that for any $\gamma \in F/\mathcal{O}_F$ we have

$$\bigcap_{\{\beta \in F/\mathcal{O}_F \mid \pi\beta = \gamma\}} N_l(\beta) = \varpi^n N_{l+1}(\gamma) \tag{1}$$

Similarly, one may define

$$M_l(\beta) = Span_{\mathcal{O}_C}\left\{\left(\varpi^{-l}\right)^{n-i}\left(u - \varpi^{-l}\beta\right)^i \mid 0 \leq i \leq n\right\}$$

This family of lattices has the nice property that

$$M_l(\beta) \subseteq M_{l+1}(\varpi\beta). \tag{2}$$

When $n < q$, it turns out that $M_l(\beta) = N_l(\beta)$. In this case, both properties combined with the recursive relations on the coefficients, suffice to establish the proof.

However, when $n \geq q$, this is no longer the case. In fact, if one begins with a family of lattices $M_l(\beta)$, we may modify it either to satisfy property (1) or property (2), call the resulting families $M_l^\sharp(\beta)$ and $M_l^\flat(\beta)$ respectively.

It turns out that if $n \geq q$ the process of performing the $\flat$ and $\sharp$ operations alternately results in modules which contain a C-line. This means that the approach taken here of using these properties combined with the recursive relations, essentially fails for $n \geq q$.

The other drawback of this approach is that the Kirillov model has such an explicit description as a space of functions only for $G = GL_2(F)$. For other reductive groups, working with the Kirillov model is much more difficult, and no longer reduces the problem purely to functional analysis. Furthermore, in the case of $G = GL_2(F)$, the

Kirillov model can be identified with a Fourier transform of the standard model (up to multiplication by a character). For other reductive groups, this is no longer the case, and when the unipotent radical of B is no longer abelian, a possible approach is to replace the Kirillov model with the non-abelian Fourier transform of the standard model, which again is much more complicated.

Large Algebraic Weights for $GL_2(F)$

Although many results were obtained for representations of $GL_2(F)$, it has been very difficult to prove the existence of $GL_2(F)$-invariant norms in locally algebraic representations of large weights, and all results known so far have severe restriction on the weights. In Chapter 3, we provide a proof of several results, establishing the existence of $GL_2(F)$-invariant norms in many unramified locally algebraic representations of large weights. We employ methods developed by Breuil in [4] and by de Ieso in [10].

Keeping notations, we let $\tau = \det(\cdot)^m \otimes Sym^n$, and write $n = d \cdot q + r$ with $0 \le r < q$. We also set $a = (\chi_1^{-1}(\varpi) + q \cdot \chi_2^{-1}(\varpi)) \cdot \varpi^{-m} \in C$. We further denote by e the ramification degree of F over $\mathbb{Q}_p$. In this case, the Breuil-Schneider conditions are equivalent simply to $a \in \mathcal{O}_C$. Then the main theorem we prove in Chapter 3 is

Theorem 15. *Assume that one of the following conditions is satisfied:*

(i) $n \le \frac{1}{2}q^2$ *with* $r < q - d$ *and* $v_F(a) \in [0, 1]$.

(ii) $n \le \frac{1}{2}q^2$ *with* $2v_F(a) - 1 \le r \le q - d$ *and* $v_F(a) \in [1, e]$.

(iii) $n \le \min\left(p \cdot q - 1, \frac{1}{2}q^2\right)$, *with* $d - 1 \le r$ *and* $v_F(a) \ge d$.

Then π *admits an integral structure.*

The proof of this theorem relies on the ideas of Breuil in [4], and it is highly technical and involved. The main idea is looking at ρ as a quotient of the universal spherical module for compact induction. This universal module has a natural integral structure, whose image under the quotient map is an excellent candidate to be an integral structure in ρ. The quotient is formed using the Hecke algebra, and the analysis is performed on the Bruhat-Tits tree of $G = GL_2(F)$.

In fact, if τ_0 is the $\mathcal{O}_C$-points of the algebraic representation τ, it can be viewed naturally as a lattice in τ, which gives rise to the lattice $ind_{KZ}^G(\tau_0)$ in $ind_{KZ}^G(\tau)$, where $K = GL_2(\mathcal{O}_F)$ is the standard maximal compact subgroup, $Z = Z(G)$ is the center of G, and ind_{KZ}^G is the functor of compact induction.

Now, the representation $ind_{KZ}^G(\tau)$ can be viewed as functions on the vertices of the Bruhat-Tits tree, taking values in τ (up to a choice of a representative at each vertex). Denote by $B_N \subseteq ind_{KZ}^G(\tau)$ the subset of functions supported on the ball of radius N around the standard chamber. Denote by T a Hecke operator generating the spherical Hecke algebra, normalized so that it is integral and $\varpi^{-1}T$ is not.

Then, following Breuil, showing that $ind_{KZ}^G(\tau_0)$ is an integral structure reduces to the following statement.

Theorem 16. *Let the assumptions be as in the previous theorem. For all large enough $N \in \mathbb{Z}_{>0}$ there exists a constant $\epsilon \in \mathbb{Z}_{\geq 0}$ depending only on N, n, a such that for all $k \in \mathbb{Z}_{\geq 0}$, and all $f \in ind_{KZ}^G(\tau)$ we have*

$$(T - a)(f) \in B_N + \varpi^k ind_{KZ}^G \tau_0 \Rightarrow f \in B_{N-1} + \varpi^{k-\epsilon} ind_{KZ}^G \tau_0.$$

Writing $f = \sum_{m=0}^M f_m$ with f_m supported on the sphere of radius $N + m$, one can show the above by decreasing induction on m, using the explicit formula defining T.

The theorem and its proof are far from satisfying. There are many artificial restrictions arising from this method of proof, since we rely on certain "miracles" such as divisibility of binomial coefficients and invertibility of certain matrices. Moreover, the proof depends (as in Breuil's original proof for $n < 2p$) on a certain choice of representatives for κ_F in $\mathcal{O}_F$, namely the Teichmüller representatives. Except from being unnatural, it means that generalization of this method to other reductive groups, where there is not always such nice choice of representatives, is far from immediate. Indeed, the unipotent radical N of the parabolic subgroup B is not as simple, and not always abelian. For example, if $G = GL_3(F)$, then N is the Heisenberg group. Therefore, we might not always have Teichmüller representatives.

Moreover, it seems that there should be a more enlightening way of establishing this result, since one may reduce the statement of the theorem to a statement about $I(1)$-invariants of the reduction mod ϖ^k, where $I(1)$ is the pro-p Iwahori subgroup corresponding to B, namely $I(1) = red^{-1}(\mathbf{N}(\kappa_F))$ with red the reduction mod ϖ. This follows from the fact that we could restate it as the injectivity of a certain map between profinite $I(1)$-modules, which is equivalent to injectivity on the $I(1)$-invariants. However, trying to establish the injectivity on the $I(1)$-invariants did not seem to be any easier, and consequently is left out of the current work.

We hope that further study of the role of the $I(1)$-invariants in the reductions mod ϖ^k for $G = GL_2(F)$ and, more generally, the role of $I(1)$-cohomology for groups of higher rank, will lead to a more complete statement of the theorem in the future.

Invariant Norms for $U_3(F)$

As many attempts were made in order to find criteria for the existence of integral structures in representations of $GL_2(F)$, where F is a finite extension of $\mathbb{Q}_p$, towards the proof of the Breuil-Schneider conjecture, which concerns the case of $GL_n(F)$, and somewhat more generally, the case of split reductive groups, very little is known about the correspondence for non-split reductive groups, in particular for the unitary group. In chapter 4, we prove the existence of invariant norms in both smooth tamely ramified principal series representations of $U_3(F)$ and in unramified locally algebraic principal series representations of $U_3(F)$ of small weight. We do so by employing the methods developed by Vigneras in [26] and by Große-Klönne in [15].

Let us state our result more precisely. Let E be a quadratic extension of F, and assume that C contains also the normal closure of E. We also set for the rest of the introduction $\varpi = \varpi_E$ a uniformizer of E (and not of F!!) and $q = q_E$ to be the cardinality of the residue field of E. Let V be a 3-dimensional vector space over E, and let $x \mapsto \overline{x}$ be the nontrivial involution in $Gal(E/F)$. Denote by θ the Hermitian form on V represented by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

with respect to the standard basis on V. Explicitly

$$(u, v) = {}^t\overline{v}\theta u.$$

Let

$$G = U_3(F) = U_3(\theta) = \left\{ g \in GL_3(E) \mid {}^t\overline{g}\theta g = \theta \right\}$$

Let B be its Borel subgroup of upper triangular matrices, and let M be the maximal (nonsplit) torus of diagonal matrices contained in it. Let $\chi : E^\times \to C^\times$ and $\chi_1 : U_1(F) \to C^\times$ be smooth characters, where $U_1(F) = \{x \in E \mid x \cdot \overline{x} = 1\}$ are the norm 1 elements. Then one can consider $\chi \otimes \chi_1$ as a character of M, and inflate it to a character of B. We let

$$\sigma = Ind_B^G(\chi \otimes \chi_1)$$

be the smooth principal series representation. Fix some $a, b \in \mathbb{Z}_{\geq 0}$ and some $d \in \mathbb{Z}$. Let τ be the irreducible algebraic representation of U_3 of highest weight

$$diag(z_1, z_2, \overline{z}_1^{-1}) \mapsto z_1^a \overline{z}_1^b \cdot \det(g)^d$$

with respect to B.

In chapter 4 we prove

Theorem 17. *Assume that either*

(i) χ, χ_1 *are unramified, and that* $a, b < p$*, or*

(ii) χ, χ_1 *are tamely ramified, and that* $a = b = 0$*.*

Then π *admits an integral structure if and only if*

$$|\varpi|^{-a-b} \leq |\chi(\varpi)| \leq \left|q^{-2}\varpi^{-a-b}\right|$$

We note that this result holds both when E/F is unramified, and when E/F is ramified. Here, in contrast with chapter 2, the results are proved by two different methods. Case (i) is proved by the method of Breuil, as in [4], while case (ii) is proved by the method Vigneras uses in [26].

Although $U_3(F)$ is rank one, so the analysis on the Bruhat-Tits tree remains quite similar to $GL_2(F)$, its Borel subgroup already has a nonabelian unipotent radical, which complicates the computations. The existence of two conjugacy classes of maximal compact subgroups, and the treatment of both ramified and unramified extensions E/F, which yield different reductions, turned this generalization to be quite more involved than it appears to be.

We comment that although some of the methods employed in this work could be generalized to groups of higher rank, we wanted to focus on small rank, as there is still much yet to understand already there.

References

[1] Laurent Berger. Représentations modulaires de $GL_2(Q_p)$ et représentations galoisiennes de dimension 2. *Astérisque*, 330:263–279, 2010.

[2] Laurent Berger and Christophe Breuil. Sur quelques représentations potentiellement cristallines de $GL_2(Q_p)$. *Astérisque*, 330:155–211, 2010.

[3] Christophe Breuil. Sur quelques représentations modulaires et p-adiques de $GL2(Q_p)$: I. *Compositio Mathematica*, 138(2):165–188, 2003.

[4] Christophe Breuil. Sur quelques representations modulaires et p-adiques de $GL_2(Q_p)$: II. *JOURNAL-INSTITUTE OF MATHEMATICS OF JUSSIEU*, 2(1):23–58, 2003.

[5] Christophe Breuil and Ariane Mézard. Multiplicités modulaires et représentations de $\mathrm{GL}_2(\mathbf{Z}_p)$ et de $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ en $\ell = p$. appendice par guy henniart. sur l'unicité des types pour gl_2. *Duke Math. J.*, 115(2):205–310, 11 2002.

[6] Christophe Breuil and Peter Schneider. First steps towards p-adic langlands functoriality. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2007(610):149–180, 2007.

[7] Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paskunas, and Sug Woo Shin. Patching and the p-adic local langlands correspondence. *arXiv preprint arXiv:1310.0831*, 2013.

[8] Pierre Colmez. Représentations de GL2 (Qp) et (ϕ, γ)-modules. *Astérisque*, 330(281):509, 2010.

[9] Pierre Colmez, Gabriel Dospinescu, and Vytautas Paskunas. The p-adic local langlands correspondence for $GL_2(Q_p)$. *arXiv preprint arXiv:1310.2235*, 2013.

[10] Marco De Ieso. Existence de normes invariantes pour GL2. *Journal of Number Theory*, 133(8):2729–2755, 2013.

[11] Bertin Diarra. Sur quelques repre´ sentations p-adiques de zp. In *Indagationes Mathematicae (Proceedings)*, volume 82, pages 481–493. Elsevier, 1979.

[12] Matthew Emerton. Local-global compatibility in the p-adic langlands programme for GL2/Q. *preprint*, 2010.

[13] Matthew Emerton et al. *p*-adic *L*-functions and unitary completions of representations of *p*-adic reductive groups. *Duke Mathematical Journal*, 130(2):353–392, 2005.

[14] Jean-Marc Fontaine. Representations l-adiques potentiellement semistables. *Astérisque*, (223):321–348, 1994.

[15] Elmar Große-Klönne. On the universal module of p-adic spherical hecke algebras. *American Journal of Mathematics*, 136(3):599–652, 2014.

[16] Michael Harris and Richard Taylor. *The Geometry and Cohomology of Some Simple Shimura Varieties.(AM-151)*, volume 151. Princeton university press, 2001.

[17] Guy Henniart. Une preuve simple des conjectures de langlands pour GL (n) sur un corps p-adique. *Inventiones mathematicae*, 139(2):439–455, 2000.

[18] Yongquan Hu. Normes invariantes et existence de filtrations admissibles. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2009(634):107–141, 2009.

[19] Herve Jacquet. Sur les représentations des groupes réductifs p-adiques. *CR Acad. Sci. Paris Sér. AB*, 280:A1271–A1272, 1975.

[20] Mark Kisin. The fontaine-mazur conjecture for GL_2. *Journal of the American Mathematical Society*, 22(3):641–690, 2009.

[21] Peter Schneider and Jeremy Teitelbaum. Banach space representations and iwasawa theory. *Israel journal of mathematics*, 127(1):359–380, 2002.

[22] Peter Schneider, Jeremy Teitelbaum, and Dipendra Prasad. U(g)-finite locally analytic representations. *Representation Theory of the American Mathematical Society*, 5(5):111–128, 2001.

[23] Peter Scholze. The local langlands correspondence for GL n over p-adic fields. *Inventiones mathematicae*, 192(3):663–715, 2013.

[24] Claus M Sorensen. A proof of the breuil-schneider conjecture in the indecomposable case. *Annals of Mathematics*, 177:1–16, 2013.

[25] Marie-France Vignéras. Correspondance de langlands semi-simple pour gl (n, f) modulo l≠ p. *Inventiones mathematicae*, 144(1):177–223, 2001.

[26] Marie-France Vignéras. A criterion for integral structures and coefficient systems on the tree of PGL (2, F). *Pure and Applied Mathematics Quarterly*, 4(4), 2008.

[27] Marie-France Vigneras et al. Banach l-adic representations of p-adic groups. *Asterisque Journal*, 327(1), 2010.

2 Kirillov models and the Breuil-Schneider conjecture for $GL_2(F)$

Eran Assaf, David Kazhdan and Ehud de Shalit

Unpublished

KIRILLOV MODELS AND THE BREUIL-SCHNEIDER CONJECTURE FOR $GL_2(F)$

ERAN ASSAF, DAVID KAZHDAN, AND EHUD DE SHALIT

Abstract. Let F be a local field of characteristic 0. The Breuil-Schneider conjecture for $GL_2(F)$ predicts which locally algebraic representations of this group admit an integral structure. We extend the methods of [K-dS12], which treated smooth representations only, to prove the conjecture for some locally algebraic representations as well.

1. Introduction

1.1. **Background.** Let F be a local field of characteristic 0 and residue characteristic p, π a fixed uniformizer of F, and q the cardinality of its residue field $\mathcal{O}_F/\pi\mathcal{O}_F$. Let E be an algebraic closure of F.

Let $\mathbf{G}$ be a reductive group over F and $G = \mathbf{G}(F)$. A *locally algebraic* representation (ρ, V_ρ) of G over E is a representation of the type

$$\rho = \tau \otimes \sigma \tag{1.1}$$

where (τ, V_τ) is (the E-points of) a finite dimensional rational representation of $\mathbf{G}$, and (σ, V_σ) is a smooth representation of G over E. An *integral structure* V_ρ^0 in V_ρ is an $\mathcal{O}_E[G]$-submodule which spans V_ρ over E, but does not contain any E-line.

If τ and σ are irreducible then ρ is irreducible as well ([P01], Theorem 1). In such a case, a non-zero $\mathcal{O}_E[G]$-submodule V_ρ^0 of V_ρ is an integral structure if and only if it is properly contained in V_ρ. Indeed, the union of all E-lines in V_ρ^0, as well as the subspace of V_ρ spanned by V_ρ^0 over E, are both $E[G]$-submodules of V_ρ. If $0 \subset V_\rho^0 \subset V_\rho$ (both inclusions being proper), the irreducibility of ρ implies that the first is 0, and the second is V_ρ.

Two integral structures in V_ρ are *commensurable* if each of them is contained in a scalar multiple of the other. In general, V_ρ need not contain an integral structure. When such an integral structure exists, it need not be unique, even up to commensurability. However, if ρ is irreducible, and an integral structure does exist, there is a unique commensurability class of *minimal integral structures*, namely the class of any cyclic $\mathcal{O}_E[G]$-module. Thus, when ρ is irreducible, to test whether integral structures exist at all, it is enough to check that for some $0 \neq v \in V_\rho$, $\mathcal{O}_E[G]v$ is not the whole of V_ρ.

The existence (and classification) of integral structures in irreducible locally algebraic representations is a natural and important question for the *p-adic local Langlands programme* (see [Br10]). When $\mathbf{G} = GL_n$, a precise conjecture for the conditions on τ and σ under which an integral structure should exist in ρ was proposed by Breuil and Schneider in [Br-Sch07], and became known as *the Breuil-Schneider conjecture*. The *necessity* of these conditions was proved there in some special cases, and by Hu [Hu09] in general. The *sufficiency* tends to be, in the words of Vigneras [V], either "obvious" or "very hard", even for GL_2.

Quite generally, if $\mathbf{G}$ is an arbitrary reductive group, the simpler σ is algebraicly, the harder the question becomes. An obvious necessary condition is for the central character of ρ to be unitary[1]. Assume therefore that this is the case. If σ is supercuspidal (its matrix coefficients are compactly supported modulo the center), the existence of an integral structure is obvious. Using global methods and the trace formula, existence of an integral structure can also be proved when σ, realized over $\mathbb{C}$ by means of some field embedding $E \hookrightarrow \mathbb{C}$, is essentially discrete series (its matrix coefficients are square integrable modulo the center)[2][So13]. In these cases, no further restrictions are imposed on τ. At the other extreme stand principal series representations, where one should impose severe restrictions on τ, and the problem becomes very hard.

We warn the reader that for arithmetic applications, the minimal integral structures in an irreducible V_ρ are often insufficient. In particular, they may be non-admissible, in the sense that their reduction modulo the maximal ideal of $\mathcal{O}_E$ is a non-admissible smooth representation over $\bar{\mathbb{F}}_q$. In such a case, even if minimal integral structures are known to exist, the existence of larger admissible integral structures is a mystery, which is resolved only in special cases, again by global methods. See [Br04].

1.2. **The main result.** We now specialize to $\mathbf{G} = GL_2$. In this case the full Breuil-Schneider conjecture is known when $F = \mathbb{Q}_p$, but only by indirect methods involving (ϕ, Γ)-modules and Galois representations. It comes as a by-product of the proof of the p-adic local Langlands correspondence ($pLLC$). This large-scale project [B-B-C] depends so far crucially on the assumption $F = \mathbb{Q}_p$. It is therefore desirable to have a *direct local proof* of the Breuil-Schneider conjecture, which does not depend on $pLLC$, and which holds for arbitrary F. As mentioned above, if σ is either supercuspidal or special, there are no restrictions on τ and integral structures are known to exist. We therefore assume that $\sigma = Ind(\chi_1, \chi_2)$ is an irreducible principal series representation.

In this work we prove the Breuil-Schneider conjecture for $GL_2(F)$ in the following cases: (1) The characters χ_1 and χ_2 are unramified, $\tau = \det(.)^m \otimes Sym^n$, and the weight is low: $n < q$ (2) The χ_i are tamely ramified, and $\tau = \det(.)^m$. The second case has been done in [K-dS12] already, but the proof presented here is somewhat cleaner.

To formulate our theorem, let χ_i be smooth characters of $F^\times$ with values in $E^\times$, and ω the unramified character[3] for which $\omega(\pi) = q^{-1}$. Let B be the Borel subgroup of upper triangular matrices in G, and consider the principal series representation

$$(V_\sigma, \sigma) = Ind_B^G(\chi_1, \chi_2). \tag{1.2}$$

[1] A character $\chi : F^\times \to E^\times$ is *unitary* if its values lie in $\mathcal{O}_E^\times$.

[2] The notion of "essentially discrete series" should be invariant under $Aut(\mathbb{C})$, hence independent of the embedding of E in $\mathbb{C}$. This is known for GL_n by the work of Bernstein-Zelevinski, and for the classical groups by Tadic.

[3] This character is usually denoted $|.|$ over $\mathbb{C}$. We will have to consider $|\omega(\pi)|$, the absolute value of q^{-1} as an element of E, and we found the notation $||\pi||$ too confusing.

This is the space of functions $f : G \to E$ for which (i)

$$f\left(\left(\begin{array}{cc} t_1 & s \\ 0 & t_2 \end{array}\right) g\right) = \chi_1(t_1)\chi_2(t_2)f(g) \tag{1.3}$$

and (ii) there exists an open subgroup $H \subset G$, depending on f, such that $f(gh) = f(g)$ for all $h \in H$. The group G acts by right translation:

$$\sigma(g)f(g^{'}) = f(g^{'}g). \tag{1.4}$$

The central character of σ is $\chi_1\chi_2$, and $Ind_B^G(\chi_1, \chi_2) \simeq Ind_B^G(\omega\chi_2, \omega^{-1}\chi_1)$, unless this representation is reducible. In fact, σ is reducible precisely when $\chi_1/\omega\chi_2 = \omega^{\pm 1}$. In this "special" case σ is indecomposable of length 2, and its irreducible constituents are a one-dimensional character and a twist of the Steinberg representation by a character. The Breuil-Schneider conjecture for a twist of Steinberg, and any τ, is known (for $GL_2(F)$, see [T93] or [V08]), however, as kindly pointed out to us by one of the referees, the case where σ is an extension of the trivial representation by the Steinberg representation (in that order) is not considered there and we are unaware of efforts made in that direction for an arbitrary τ when $F \neq \mathbb{Q}_p$. Nevertheless, we exclude this case from now on, and assume that σ is *irreducible*.

Next, fix integers m and $n \geq 0$, and consider the rational representation

$$(V_\tau, \tau) = \det(.)^m \otimes Sym^n, \tag{1.5}$$

where Sym^n denotes the nth symmetric power of the standard representation of $\mathbf{G}$. Put

$$\begin{aligned} \lambda &= \chi_1(\pi),\ \mu = \omega\chi_2(\pi), \\ \tilde{\lambda} &= \lambda\pi^m,\ \tilde{\mu} = \mu\pi^m. \end{aligned} \tag{1.6}$$

The Breuil-Schneider conjecture for $\rho = \tau \otimes \sigma$ predicts that ρ has an integral structure if and only if the following two conditions are satisfied:

$$\text{(i) } |\tilde{\lambda}\tilde{\mu}q\pi^n| = 1 \quad \text{(ii)} |\tilde{\lambda}| \leq |q^{-1}\pi^{-n}|,\ |\tilde{\mu}| \leq |q^{-1}\pi^{-n}|. \tag{1.7}$$

Condition (i) means that the central character of ρ is unitary. Given (i), (ii) is equivalent to $1 \leq |\tilde{\lambda}| \leq |q^{-1}\pi^{-n}|$ or to the symmetric condition for $\tilde{\mu}$. It is known (and easy to prove) that these two conditions are necessary.

Theorem 1.1. *Assume that (i) and (ii) are satisfied. Assume, in addition, that* either *(1)* χ_1 *and* χ_2 *are unramified and* $n < q$, or *(2) that* χ_1 *and* χ_2 *are tamely ramified and* $n = 0$. *Then* ρ *has an integral structure.*

Although our method is new, and gives some new insight into the minimal integral structure (see Theorem 1.2 below), the two cases have been known before: case (1) by Breuil [Br03] (for $\mathbb{Q}_p$) and de Ieso [dI12] (for general F), and case (2) by Vigneras [V08]. It is interesting to note that the restriction $n < q$ in case (1) and the restriction on tame ramification in case (2) are also needed in the above mentioned works. In fact, Breuil, de Ieso and Vigneras all use, in one way or another, the method of *compact induction*, replacing the representation ρ by a local system on the tree of G. Our approach takes place in a certain *dual* space of functions on F. Any attempt to translate it to the set-up of the tree involves the p-adic Fourier transform, which is unbounded, and makes it impossible to trace back the arguments. The way in which the weight and ramification restrictions are brought to bear on the problem are also not similar, yet the very same restrictions turn out to be necessary for the proofs to work.

1.3. **An outline of the proof.** As in [K-dS12], our approach is based on a study of the Kirillov model of ρ. For the sake of exposition we now exclude the case $\chi_1 = \omega\chi_2$, which requires special attention. Assuming $\chi_1 \neq \omega\chi_2$, the *Kirillov model* of ρ is then the following space of functions on $F - \{0\}$:

$$\mathcal{K} = C_c^\infty(F,\tau)\chi_1 + C_c^\infty(F,\tau)\omega\chi_2 \subseteq C^\infty(F^\times,\tau). \tag{1.8}$$

Here $C_c^\infty(F,\tau)$ is the space of V_τ-valued locally constant functions of compact support on F. The model $\mathcal{K}$ is obtained by tensoring τ with the classical Kirillov model of the smooth representation σ (see [Bu98]). It contains $\mathcal{K}_0 = C_c^\infty(F^\times,\tau)$, the subspace of functions vanishing near 0, and $\mathcal{K}/\mathcal{K}_0$ consists of two copies of V_τ. When $\tau = 1$, this is just the *Jacquet module* of $\mathcal{K}$. The characters χ_1 and $\omega\chi_2$ are the *exponents* of the Jacquet module, the two characters by which the torus of diagonal matrices acts on it.

We record the action of an element

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in B \tag{1.9}$$

on $\phi \in \mathcal{K}$. Fix an additive character $\psi : F \to E^\times$ under which $\mathcal{O}_F$ is its own annihilator for the pairing $(\beta, x) \mapsto \psi(\beta x)$. Then

$$\rho(g)\phi(x) = \tau(g)\left(\psi(bx)\phi(ax)\right). \tag{1.10}$$

The action of G in the model $\mathcal{K}$ depends on the choice of ψ, but only up to isomorphism.

At this point, we must introduce more notation and recall some easy facts. Let $U_F = \mathcal{O}_F^\times$ be the group of units in $\mathcal{O}_F$. Let 1_S be the characteristic function of $S \subset F$, and $\phi_l = 1_{\pi^l U_F}$ ($l \in \mathbb{Z}$). If $b \in F$, write $\psi_b(x) = \psi(bx)$. The function $\psi_b(\pi^{-l}x)\phi_l(x)$ depends only on β, the image of b in $W = F/\mathcal{O}_F$, so from now on we denote it by $\psi_\beta(\pi^{-l}x)\phi_l(x)$. Any locally constant function on the annulus $\pi^l U_F$ can be expanded as a finite linear combination of these functions. Moreover, Fourier analysis on the disk $\pi^l\mathcal{O}_F$ implies that

$$\sum_{\beta\in W} C_l(\beta)\psi_\beta(\pi^{-l}x)\phi_l(x) = 0 \tag{1.11}$$

if and only if $C_l(\beta)$ depends only on $\pi\beta$, i.e.

$$C_l(\beta) = C_l(\beta') \text{ if } \beta - \beta' \in W_1 = \pi^{-1}\mathcal{O}_F/\mathcal{O}_F. \tag{1.12}$$

The same applies of course to V_τ-valued functions, except that now the coefficients $C_l(\beta) \in V_\tau$.

An arbitrary function $\phi \in \mathcal{K}$ may be expanded annulus-by-annulus as

$$\phi = \sum_{l=l_0}^{\infty} \sum_{\beta\in W} C_l(\beta)\psi_\beta(\pi^{-l}x)\phi_l(x), \tag{1.13}$$

where $C_l(\beta) \in V_\tau$, $l_0 \in \mathbb{Z}$ is the valuation of the outermost annulus on which ϕ is supported, and for every l only finitely many $C_l(\beta) \neq 0$. The only restriction on ϕ is imposed by the asymptotics as $x \to 0$. In particular, finite linear combinations as above represent the elements of $\mathcal{K}_0$. One should think of the β as frequencies, and of the $C_l(\beta)$ as the amplitudes attached to these frequencies on the annulus $\pi^l U_F$. These amplitudes are not uniquely defined since we may add to $C_l(\beta)$ a perturbation $\tilde{C}_l(\beta)$ without affecting $\phi|\pi^l U_F$, provided $\tilde{C}_l(\beta) = \tilde{C}_l(\beta^{'})$ whenever $\beta - \beta^{'} \in W_1$. But as explained above, this is the only ambiguity.

Theorem 1.1 follows from the following more precise result, which makes the integral structure on V_ρ "visible".

Theorem 1.2. *Let the assumptions be as in Theorem 1.1. Let V_ρ^0 be the $\mathcal{O}_E[G]$-submodule of $V_\rho = \mathcal{K}$ spanned by a non-zero vector. Then there exist $\mathcal{O}_E$-lattices $M_0(\beta) \subset V_\tau$ such that if $\phi \in V_\rho^0$ vanishes outside $\mathcal{O}_F$, it has an expansion as above with $C_0(\beta) \in M_0(\beta)$ for every β.*

Note that we do not claim that the values of $\phi \in V_\rho^0$ are bounded on U_F, nor at any other point. The amplitudes can be bounded only separately, and only on the first annulus where ϕ does not vanish. Since the $C_0(\beta)$ are not uniquely defined, one still needs a simple argument to show that this is good enough.

Proposition 1.3. *Theorem 1.2 implies Theorem 1.1.*

Proof. We shall show that $V_\rho^0 \neq V_\rho$, so in view of the irreducibility of ρ, V_ρ^0 will be an integral structure. Consider the function $\phi = C\phi_0$ where $C \in V_\tau$ *lies outside* $M = \sum_{\beta \in W_1} M_0(\beta)$. Suppose, by way of contradiction, that $\phi \in V_\rho^0$. Then ϕ is also given by an expansion as in Theorem 1.2. For $x \in U_F$ we must have then

$$C = \sum_{\beta \in W} C_0(\beta)\psi_\beta(x). \tag{1.14}$$

This forces, as we have seen, the equality $C_0(0) - C = C_0(\beta)$ for $\beta \in W_1 - \{0\}$. But this contradicts the choice of C. □

We now make some comments on the proof of Theorem 1.2. The first step is standard. Using the decomposition $G = BK$, $K = GL_2(\mathcal{O}_F)$, we show that V_ρ^0 is commensurable with a certain $\mathcal{O}_E[B]$-module of finite type Λ which also spans V_ρ over E. We may therefore prove the assertion of the theorem for Λ instead of V_ρ^0. Our Λ will be spanned over $\mathcal{O}_E$ by an explicit infinite set $\mathcal{E}$ of nice functions.

Pick a $\phi \in \Lambda$, express it as a linear combination of the functions in $\mathcal{E}$, and expand it annulus-by-annulus as above. The coefficients $C_l(\beta)$ then satisfy *recursive relations*, in which the coefficients used to express ϕ as a linear combination of $\mathcal{E}$ figure out.

Suppose that ϕ vanishes off $\mathcal{O}_F$. It may still be the case that $C_l(\beta) \neq 0$ for some β and $l < 0$. However, cancellation must take place, and as we have seen, $C_l(\beta)$ depends then, for $l < 0$, on $\pi\beta$ only. We proceed by increasing induction on l and show that $C_l(\beta)$ must belong, for $l \leq 0$, to a certain $\mathcal{O}_E$-lattice $M_l(\beta) \subset V_\tau$, depending on l and β, but not on ϕ. When $l = 0$ we reach the desired conclusion.

Two phenomena assist us in establishing these bounds on the coefficients. The first, which has already been utilized in our previous work [K-dS12], is that in the recursive relations for $C_l(\beta)$ we encounter terms such as

$$\sum_{\pi\alpha = \beta} C_{l-1}(\alpha). \tag{1.15}$$

As long as $l \leq 0$, the q summands are all equal, so their sum is equal to $qC_{l-1}(\alpha_\beta)$, where α_β is any one of the α's. The factor q is small, and helps to control $C_l(\beta)$.

The second phenomenon is new, and more subtle. The information that $C_l(\beta)$ depends only on $\pi\beta$, puts a further restriction on $C_l(\beta)$, beyond lying in $M_l(\beta)$, which is vital for the deduction that the $C_{l+1}(\gamma)$ lie in $M_{l+1}(\gamma)$. For example, assume that $m = 0$ and $n = 1$, so τ is the standard representation of G on E^2, and let e_1 and e_2 be the standard basis. In this example, up to scaling,

$$M_l(\beta) = Span_{\mathcal{O}_E}\left\{\pi^{-l}e_1, e_2 - \pi^{-l}\beta e_1\right\} \tag{1.16}$$

(note that this is indeed well defined, i.e. depends only on $\beta \bmod \mathcal{O}_F$). It is easily checked that if $C_l(\beta) \in M_l(\beta)$ for all β, and *in addition,* $C_l(\beta)$ depends only on $\pi\beta$, then in fact

$$C_l(\beta) \in Span_{\mathcal{O}_E}\left\{\pi^{-l}e_1, \pi(e_2 - \pi^{-l}\beta e_1)\right\}. \tag{1.17}$$

This minor improvement on $C_l(\beta) \in M_l$ is crucial for our method to work. Roughly speaking, the first phenomenon described above takes care of the factor q^{-1} in condition (1.7)(ii), while the second one takes care of the π^{-n}.

The inductive procedure requires also the relation $M_l(\beta) \subset M_{l+1}(\pi\beta)$. It is here that we need the condition $n < q$. We may modify the definition of $M_l(\beta)$ to guarantee this relation without any restriction on n, but we then lose the subtle phenomenon to which we alluded in the previous paragraph. At present, we are unable to hold the rope at both ends simultaneously.

When χ_1 and χ_2 are unramified this is the end of the story. When χ_1 and χ_2 are ramified, two types of complications occur. First, we must give up the algebraic part τ (except for the benign twist by the determinant). Second, in the recursive relations used to define $C_l(\beta)$, Gauss sums intervene. These Gauss sums have denominators which are still under control if the characters are only tamely ramified, but if the χ_i are wildly ramified, our method breaks down. It is interesting to note that the well-known estimates on Gauss sums intervene also in Vigneras' proof of the tamely-ramified smooth case of the conjecture.

In the remaining cases, not covered by (1) or (2), it is possible that Theorem 1.2 fails, yet Theorem 1.1 continues to hold, for a different reason. It will be interesting to check numerically whether one should expect Theorem 1.2 in general. Even for $F = \mathbb{Q}_p$, where, as mentioned above, the full conjecture is known, it is unclear to us whether Theorem 1.2 holds beyond cases (1) and (2).

2. Preliminary results

2.1. **Fourier analysis on $\mathcal{O}_F$.** The discrete group $W = F/\mathcal{O}_F$ is the topological dual of $\mathcal{O}_F$ via the pairing

$$(\beta, x) \mapsto \psi_\beta(x) = \psi(\beta x). \tag{2.1}$$

Every locally constant E-valued function on $\mathcal{O}_F$ has a unique finite Fourier expansion

$$\phi = \sum_{\beta \in W} c(\beta)\psi_\beta(x). \tag{2.2}$$

The proof of the following easy lemma is left to the reader.

Lemma 2.1. *(i) $\phi|U_F = 0$ if and only if $c(\beta)$ depends only on $\pi\beta$. (ii) $\phi|\pi\mathcal{O}_F = 0$ if and only if $\sum_{\pi\beta=\gamma} c(\beta) = 0$ for every $\gamma \in W$.*

The lemma is immediately translated to a similar one in the disk $\pi^l\mathcal{O}_F$ using the functions $\psi_\beta(\pi^{-l}x)$ as a basis for the expansion.

2.2. **Lattices in V_τ.** If $\beta \in W$ and $l \in \mathbb{Z}$ let

$$D_l(\beta) = \left\{ \alpha \in F \,|\, |\alpha - \pi^{-l}\beta| \le |\pi^{-l}| \right\}. \tag{2.3}$$

This disk indeed depends only on $\beta \bmod \mathcal{O}_F$. Note that

$$D_{l+1}(\gamma) = \coprod_{\pi\beta=\gamma} D_l(\beta). \tag{2.4}$$

Let $\tau = \det(.)^m \otimes Sym^n$. Identify V_τ with $E[u]^{\le n}$, the space of polynomials of degree at most n, with the action

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) u^i = (ad-bc)^m (a+cu)^{n-i}(b+du)^i. \tag{2.5}$$

Let

$$N_l(\beta) = \left\{ P \in V_\tau \,|\, |P(\alpha)| \le |\pi|^{-nl} \ \forall \alpha \in D_l(\beta) \right\}. \tag{2.6}$$

These are lattices in V_τ.

Lemma 2.2. *(i) For any $\gamma \in W$*

$$\bigcap_{\pi\beta=\gamma} N_l(\beta) = \pi^n N_{l+1}(\gamma). \tag{2.7}$$

(ii) Assume that $n < q$. Then

$$N_l(\beta) = Span_{\mathcal{O}_E}\left\{ (\pi^{-l})^{n-i}(u - \pi^{-l}\beta)^i \quad (0 \le i \le n) \right\}. \tag{2.8}$$

(iii) Assume that $n < q$. Then

$$N_l(\beta) \subset N_{l+1}(\pi\beta). \tag{2.9}$$

Proof. (i) If $P \in N_l(\beta)$ then it is bounded by $|\pi|^{-nl}$ on $D_l(\beta)$. But the q disks $D_l(\beta)$, for the β satisfying $\pi\beta = \gamma$, cover $D_{l+1}(\gamma)$. The result follows.

(ii) Clearly $P \in N_l(\beta)$ if and only if $\pi^{nl}P(\pi^{-l}u + \pi^{-l}\beta) \in N_0(0)$. It is therefore enough to prove that $|P(\alpha)| \le 1$ for all $\alpha \in \mathcal{O}_F$ if and only if $P \in \mathcal{O}_E[u]^{\le n}$. This is well-known, but note that it fails if $n \ge q$ (consider $\pi^{-1}(u^q - u)$).

(iii) This is an immediate consequence of (ii).

□

2.3. **Passing from $\mathcal{O}_E[B]$-modules to $\mathcal{O}_E[G]$-modules.** Consider the representation V_ρ, where $\rho = \tau \otimes \sigma$, $\tau = \det(.)^m \otimes Sym^n$, and $\sigma = Ind_B^G(\chi_1, \chi_2)$ are as in the introduction.

Proposition 2.3. *Let $v_1, \ldots, v_r \in V_\sigma$ be such that the module $\Lambda_\sigma = \sum_{j=1}^r \mathcal{O}_E[B]v_j$ spans V_σ over E. Let*

$$\Lambda = \sum_{i=0}^n \sum_{j=1}^r \mathcal{O}_E[B]\left(u^i \otimes v_j\right) \subset V_\rho. \tag{2.10}$$

Then Λ is commensurable with every cyclic $\mathcal{O}_E[G]$-submodule of V_ρ.

Proof. Let $K = GL_2(\mathcal{O}_F)$ and recall that $G = BK$. If $N \le K$ is a subgroup of finite index fixing all the v_j, then N preserves the finitely generated $\mathcal{O}_E$-submodule

$$\sum_{i,j} \mathcal{O}_E(u^i \otimes v_j), \tag{2.11}$$

because $\tau(K)$ preserves $\mathcal{O}_E[u]^{\le n}$. It follows that $\sum_{i,j}\mathcal{O}_E[K](u^i\otimes v_j)$ is finitely generated over $\mathcal{O}_E$. Since Λ spans V_ρ over E, there is a constant $c\in E$ such that

$$\sum_{i,j}\mathcal{O}_E[K](u^i\otimes v_j)\subset c\Lambda. \tag{2.12}$$

But then

$$\begin{aligned}\sum_{i,j}\mathcal{O}_E[G](u^i\otimes v_j) &= \mathcal{O}_E[B]\sum_{i,j}\mathcal{O}_E[K](u^i\otimes v_j)\\ &\subset \mathcal{O}_E[B](c\Lambda)=c\Lambda.\end{aligned} \tag{2.13}$$

On the other hand, $\Lambda\subset\sum_{i,j}\mathcal{O}_E[G](u^i\otimes v_j)$. The two inclusions prove the proposition, since the sum of a finite number of cyclic modules, all being commensurable, is again commensurable with any cyclic module.

□

Corollary 2.4. *To prove Theorem 1.2 we may replace V_ρ^0 by Λ.*

2.4. **The Kirillov model and a choice of Λ.** Assume from now on that $\chi_1\neq\omega\chi_2$. The exceptional case $\chi_1=\omega\chi_2$ requires special attention and will be dealt with in the end. Let $\mathcal{K}$ be the model of V_ρ described in the introduction. For $\{v_j\}$ we choose the two functions

$$v_1=F_0^{'}(x)=1_{\mathcal{O}_F}\chi_1,\quad v_2=F_0^{''}=1_{\mathcal{O}_F}\omega\chi_2. \tag{2.14}$$

Let $F_k^{'}(x)=F_0^{'}(\pi^{-k}x)$ and similarly $F_k^{''}(x)=F_0^{''}(\pi^{-k}x)$. Since

$$\sigma\left(\begin{pmatrix}\pi^{-k} & -\pi^{-k}\beta\\ & 1\end{pmatrix}\right)F_0^{'}(x)=\psi_\beta(-\pi^{-k}x)F_k^{'}(x) \tag{2.15}$$

and similarly for $F_0^{''}(x)$, we see that $\Lambda_\sigma=\mathcal{O}_E[B]F_0^{'}+\mathcal{O}_E[B]F_0^{''}$ spans V_σ over E.

Lemma 2.5. *Let $\Lambda=\sum_{i=0}^n\sum_{j=1}^2\mathcal{O}_E[B]\left(u^i\otimes v_j\right)$, where $v_1=F_0^{'}$ and $v_2=F_0^{''}$. Then every element of Λ can be written as a finite sum*

$$\phi=\sum_{k=k_0}^{\infty}\sum_{\beta\in W}c_k^{'}(\beta)\psi_\beta(-\pi^{-k}x)F_k^{'}(x)+c_k^{''}(\beta)\psi_\beta(-\pi^{-k}x)F_k^{''}(x), \tag{2.16}$$

where $c_k^{'}(\beta),c_k^{''}(\beta)\in\pi^{-km}N_k(\beta)$., and $k_0\in\mathbb{Z}$ is the minimal k for which the coefficients are nonzero.

Proof. Since the central character of ρ is unitary (condition (1.7)(i)), it is enough to span Λ by matrices in the mirabolic subgroup

$$\left\{\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\right\}\le B. \tag{2.17}$$

Furthermore, as $B\cap K$ stabilizes $\sum_{i=0}^n\sum_{j=1}^2\mathcal{O}_E\left(u^i\otimes v_j\right)$, we see that

$$\begin{aligned}\Lambda &= \sum_{k\in\mathbb{Z}}\sum_{\beta\in W}\mathcal{O}_E\rho\left(\begin{pmatrix}\pi^{-k} & -\pi^{-k}\beta\\ & 1\end{pmatrix}\right)\left(u^i\otimes v_j\right)\\ &= \sum_{k\in\mathbb{Z}}\sum_{\beta\in W}\pi^{-km}(\pi^{-k})^{n-i}(u-\pi^{-k}\beta)^i\otimes\psi_\beta(-\pi^{-k}x)\left(\mathcal{O}_EF_k^{'}(x)+\mathcal{O}_EF_k^{''}(x)\right).\end{aligned}$$

The coefficients $(\pi^{-k})^{n-i}(u-\pi^{-k}\beta)^i\in N_k(\beta)$, see Lemma 2.2(ii).

□

3. The unramified case

3.1. The recursion relations. Assume now that χ_1 and χ_2 are unramified. Recall that $\lambda = \chi_1(\pi)$, $\mu = \omega\chi_2(\pi)$, and $\phi_l = 1_{\pi^l U_F}$. Then

$$F_k^{'}(x) = \sum_{l=k}^{\infty} \lambda^{l-k}\phi_l, \quad F_k^{''}(x) = \sum_{l=k}^{\infty} \mu^{l-k}\phi_l. \tag{3.1}$$

Pick a $\phi \in \Lambda$. Substituting (3.1) in the expression (2.16), and rearranging the sum "by annuli" we get

$$\phi = \sum_{l=k_0}^{\infty} \sum_{\beta \in W} C_l(\beta)\psi_\beta(-\pi^{-l}x)\phi_l(x), \tag{3.2}$$

where

$$\begin{aligned} C_l(\beta) &= C_l^{'}(\beta) + C_l^{''}(\beta), \\ C_l^{'}(\beta) &= \sum_{k=k_0}^{l} \lambda^{l-k} \sum_{\pi^{l-k}\alpha=\beta} c_k^{'}(\alpha), \\ C_l^{''}(\beta) &= \sum_{k=k_0}^{l} \mu^{l-k} \sum_{\pi^{l-k}\alpha=\beta} c_k^{''}(\alpha). \end{aligned} \tag{3.3}$$

We deduce that

$$\begin{aligned} C_{k_0}^{'}(\beta) &= c_{k_0}^{'}(\beta) \\ C_l^{'}(\beta) &= \lambda \sum_{\pi\alpha=\beta} C_{l-1}^{'}(\alpha) + c_l^{'}(\beta), \end{aligned} \tag{3.4}$$

and similarly for $C_l^{''}(\beta)$, with μ instead of λ. We now derive from these relations a recursion relation for the $C_l(\beta)$, going two generations backwards.

Lemma 3.1. *Let $c_l = c_l^{'} + c_l^{''}$. Then $C_{k_0}(\beta) = c_{k_0}(\beta)$ and*

$$\begin{aligned} C_{l+1}(\gamma) &= (\lambda+\mu)\sum_{\pi\beta=\gamma} C_l(\beta) - \mu\lambda \sum_{\pi\beta=\gamma}\sum_{\pi\alpha=\beta} C_{l-1}(\alpha) \\ &\quad - \sum_{\pi\beta=\gamma} (\lambda c_l^{''}(\beta) + \mu c_l^{'}(\beta)) + c_{l+1}(\gamma). \end{aligned} \tag{3.5}$$

Proof. We add the relations that we have obtained for $C_l^{'}(\beta)$ and $C_l^{''}(\beta)$ and rearrange them. We do the same at level $l+1$. Letting α, β and γ range over W as usual, we get

$$\begin{aligned} C_l(\beta) &= \lambda \sum_{\pi\alpha=\beta} C_{l-1}(\alpha) + (\mu-\lambda)\sum_{\pi\alpha=\beta} C_{l-1}^{''}(\alpha) + c_l(\beta), \\ C_{l+1}(\gamma) &= \lambda \sum_{\pi\beta=\gamma} C_l(\beta) + (\mu-\lambda)\sum_{\pi\beta=\gamma} C_l^{''}(\beta) + c_{l+1}(\gamma). \end{aligned} \tag{3.6}$$

To deal with the middle term in the *second* equation we use the recursive relation for $C_l^{''}(\beta)$ and then eliminate $(\mu-\lambda)\sum_{\pi\alpha=\beta} C_{l-1}^{''}(\alpha)$ using the *first* equation:

$$\begin{aligned}(\mu-\lambda)\sum_{\pi\beta=\gamma} C_l^{''}(\beta) &= (\mu-\lambda)\sum_{\pi\beta=\gamma}\left(\mu\sum_{\pi\alpha=\beta} C_{l-1}^{''}(\alpha)+c_l^{''}(\beta)\right)\\ &= \mu\sum_{\pi\beta=\gamma}\left(C_l(\beta)-\lambda\sum_{\pi\alpha=\beta}C_{l-1}(\alpha)-c_l(\beta)\right)\\ &\quad +(\mu-\lambda)\sum_{\pi\beta=\gamma}c_l^{''}(\beta)\\ &= \mu\sum_{\pi\beta=\gamma}C_l(\beta)-\mu\lambda\sum_{\pi\beta=\gamma}\sum_{\pi\alpha=\beta}C_{l-1}(\alpha)\\ &\quad -\sum_{\pi\beta=\gamma}(\lambda c_l^{''}(\beta)+\mu c_l^{'}(\beta)). \end{aligned} \tag{3.7}$$

The lemma follows from this.

□

3.2. **Conclusion of the proof.** Let ρ satisfy the conditions of Theorem 1.2, i.e. the estimates (1.7)(i) and (ii) on λ and μ, and $n<q$. Pick a $\phi\in\Lambda$ as before, and expand it as in (3.2). Assume that it vanishes outside of $\mathcal{O}_F$. Let

$$M_l(\beta)=q^{-1}\pi^{-n-lm}N_l(\beta). \tag{3.8}$$

Lemma 3.2. *For every $k_0\le l\le 0$ and every $\beta\in W$, $C_l(\beta)\in M_l(\beta)$.*

Proof. We apply Lemma 2.2 and Lemma 2.5, and prove the desired bound on $C_l(\beta)$ by increasing induction on l.

When $l=k_0$, $C_{k_0}(\beta)=c_{k_0}(\beta)\in\pi^{-k_0m}N_{k_0}(\beta)\subset M_{k_0}(\beta)$. Suppose that the lemma has been established up to index l, and $l+1\le 0$. Then $C_l(\beta)$ (resp. $C_{l-1}(\alpha)$) depends only on $\pi\beta$(resp. $\pi\alpha$), since ϕ vanishes on $F-\mathcal{O}_F$. We invoke the recursion relation (3.5) for $C_{l+1}(\gamma)$. The term

$$\sum_{\pi\beta=\gamma}(\lambda c_l^{''}(\beta)+\mu c_l^{'}(\beta))\in M_{l+1}(\gamma) \tag{3.9}$$

since $c_l^{'}(\beta), c_l^{''}(\beta)\in\pi^{-lm}N_l(\beta)$, $|\mu|,|\lambda|\le|q^{-1}\pi^{-n-m}|$, and because of the relation $N_l(\beta)\subset N_{l+1}(\gamma)$, that holds whenever $\pi\beta=\gamma$. That

$$c_{l+1}(\gamma)\in M_{l+1}(\gamma) \tag{3.10}$$

is clear. The term

$$(\lambda+\mu)\sum_{\pi\beta=\gamma}C_l(\beta)\in M_{l+1}(\gamma) \tag{3.11}$$

because the q summands $C_l(\beta)$ are *equal*, hence belong to

$$\bigcap_{\pi\beta=\gamma}M_l(\beta)=q^{-1}\pi^{-n-lm}\bigcap_{\pi\beta=\gamma}N_l(\beta)=q^{-1}\pi^{-lm}N_{l+1}(\gamma). \tag{3.12}$$

Thus $\sum_{\pi\beta=\gamma}C_l(\beta)\in\pi^{-lm}N_{l+1}(\gamma)$, while $|\lambda+\mu|\le|q^{-1}\pi^{-n-m}|$. Finally,

$$\mu\lambda\sum_{\pi\beta=\gamma}\sum_{\pi\alpha=\beta}C_{l-1}(\alpha)\in M_{l+1}(\gamma) \tag{3.13}$$

for similar reasons: For a given β, the q summands $C_{l-1}(\alpha)$ are equal, so belong to

$$\bigcap_{\pi\alpha=\beta} M_{l-1}(\alpha) = q^{-1}\pi^{-n-(l-1)m} \bigcap_{\pi\alpha=\beta} N_{l-1}(\alpha) = q^{-1}\pi^{-(l-1)m} N_l(\beta). \tag{3.14}$$

This implies that their sum, $\sum_{\pi\alpha=\beta} C_{l-1}(\alpha) \in \pi^{-(l-1)m} N_l(\beta) \subset \pi^{-(l-1)m} N_{l+1}(\gamma)$. But $|\mu\lambda| = |q^{-1}\pi^{-n-2m}|$, so for every β

$$\mu\lambda \sum_{\pi\alpha=\beta} C_{l-1}(\alpha) \in q^{-1}\pi^{-n-(l+1)m} N_{l+1}(\gamma) = M_{l+1}(\gamma). \tag{3.15}$$

Since each of the four terms in (3.6) has been shown to lie in $M_{l+1}(\gamma)$, the proof of the induction step is complete.

□

When $l = 0$, $C_0(\beta) \in M_0(\beta)$, and this proves Theorem 1.2.

4. The case $\chi_1 = \omega\chi_2$

We finally deal with the one excluded case, when $\chi_1 = \omega\chi_2$. After a twist by a character of finite order we may assume that χ_1 is unramified. In this case $\lambda = \mu$ and the Kirillov model is the space

$$\mathcal{K} = C_c^\infty(F, \tau)\chi_1 + C_c^\infty(F, \tau)v\chi_1, \tag{4.1}$$

where $v : F^\times \to \mathbb{Z} \subset E$ is the normalized valuation. The action of B is still given by (1.10). Once more, $\mathcal{K}$ contains $\mathcal{K}_0 = C_c^\infty(F^\times, \tau)$ as a subspace. When $\tau = 1$, the quotient $\mathcal{K}/\mathcal{K}_0$ is the Jacquet module. The torus acts on it non-semisimply, by

$$\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mapsto \chi_1(t_1 t_2) \begin{pmatrix} 1 & v(t_1/t_2) \\ & 1 \end{pmatrix}. \tag{4.2}$$

Following the notation of Section 3, we let

$$F_0^{'} = \chi_1 1_{\mathcal{O}_F}, \quad F_0^{''} = -v\chi_1 1_{\mathcal{O}_F} \tag{4.3}$$

and

$$F_k^{'} = \sum_{l=k}^{\infty} \lambda^{l-k}\phi_l, \quad F_k^{''} = \sum_{l=k}^{\infty}(k-l)\lambda^{l-k}\phi_l. \tag{4.4}$$

The module Λ consists of all the functions ϕ as in (2.16), and any such ϕ can be expanded "by annuli" as in (3.2). The coefficients of the expansion are given by (3.3), except that the last equation now takes the shape

$$C_l^{''}(\beta) = \sum_{k=k_0}^{l} (k-l)\lambda^{l-k} \sum_{\pi^{l-k}\alpha=\beta} c_k^{''}(\alpha). \tag{4.5}$$

The recursion relation for $C_l^{'}(\beta)$ is given by (3.4) but $C_l^{''}(\beta)$ needs a modification.

Lemma 4.1. *We have* $C_{k_0}^{''}(\beta) = 0$, $C_{k_0+1}^{''}(\beta) = -\lambda \sum_{\pi\alpha=\beta} c_{k_0}^{''}(\alpha)$, *and for* $l > k_0$

$$C_{l+1}^{''}(\gamma) = 2\lambda \sum_{\pi\beta=\gamma} C_l^{''}(\beta) - \lambda^2 \sum_{\pi\beta=\gamma}\sum_{\pi\alpha=\beta} C_{l-1}^{''}(\alpha) - \lambda \sum_{\pi\beta=\gamma} c_l^{''}(\beta). \tag{4.6}$$

Proof. A straightforward exercise.

□

Lemma 4.2. *The following recursion relation holds:*
(4.7)
$$C_{l+1}(\gamma)=2\lambda\sum_{\pi\beta=\gamma}C_l(\beta)-\lambda^2\sum_{\pi\beta=\gamma}\sum_{\pi\alpha=\beta}C_{l-1}(\alpha)-\lambda\sum_{\pi\beta=\gamma}(c_l^{''}(\beta)+c_l^{'}(\beta))+c_{l+1}^{'}(\gamma).$$

Proof. We write

$$\begin{aligned}
C_{l+1}^{'}(\gamma) &= \lambda\sum_{\pi\beta=\gamma}C_l^{'}(\beta)+c_{l+1}^{'}(\gamma)\\
&= 2\lambda\sum_{\pi\beta=\gamma}C_l^{'}(\beta)-\lambda\sum_{\pi\beta=\gamma}\left(\lambda\sum_{\pi\alpha=\beta}C_{l-1}^{'}(\alpha)+c_l^{'}(\beta)\right)+c_{l+1}^{'}(\gamma)\\
&= 2\lambda\sum_{\pi\beta=\gamma}C_l^{'}(\beta)-\lambda^2\sum_{\pi\beta=\gamma}\sum_{\pi\alpha=\beta}C_{l-1}^{'}(\alpha)-\lambda\sum_{\pi\beta=\gamma}c_l^{'}(\beta)+c_{l+1}^{'}(\gamma) \qquad (4.8)
\end{aligned}$$

and we add the result to the recursive relation for $C_{l+1}^{''}(\gamma)$.

□

Note the similarity with Lemma 3.1. The rest of the proof of Theorem 1.2 is now identical to that given in the case $\lambda\neq\mu$ in Section 3.2.

5. The tamely ramified case

For the sake of completeness we treat also case (2) of the theorem, which is covered by [K-dS12]. The proof is the same, except that we have cleaned up the computations.

5.1. **The recursion relations.** Assume from now on that at least one of the characters χ_1 and χ_2 is ramified, but $\tau=\det(.)^m$, i.e. $n=0$. Since a twist of ρ by a character of finite order does not affect the validity of Theorem 1.2, we may assume that χ_2 is unramified. We let ε be the restriction of χ_1 to U_F, and extend it to a character of $F^\times$ so that $\varepsilon(\pi)=1$. We denote by $\nu\geq 1$ the conductor of ε. Letting $\lambda=\chi_1(\pi)$ and $\mu=\omega\chi_2(\pi)$ as before, we have

$$\chi_1(u\pi^k)=\varepsilon(u)\lambda^k,\omega\chi_2(u\pi^k)=\mu^k \qquad (5.1)$$

if $u\in U_F$.

Recall that

$$F_k^{'}=\varepsilon\sum_{l=k}^{\infty}\lambda^{l-k}\phi_l,\quad F_k^{''}=\sum_{l=k}^{\infty}\mu^{l-k}\phi_l. \qquad (5.2)$$

The module Λ consists this time of functions of the form

$$\begin{aligned}
\phi(x) &= \sum_{k=k_0}^{\infty}\sum_{\beta\in W}c_k^{'}(\beta)\psi_\beta(-\pi^{-k}x)F_k^{'}(x)+c_k^{''}(\beta)\psi_\beta(-\pi^{-k}x)F_k^{''}(x) \qquad (5.3)\\
&= \sum_{l=k_0}^{\infty}\sum_{\beta\in W}C_l(\beta)\psi_\beta(-\pi^{-l}x)\phi_l(x),
\end{aligned}$$

with $c_k^{'}(\beta), c_k^{''}(\beta) \in \pi^{-mk}\mathcal{O}_E$, and some $C_l(\beta)$ which we are now going to compute. Let, as before

$$\begin{aligned} C_l^{'}(\beta) &= \sum_{k=k_0}^{l} \lambda^{l-k} \sum_{\pi^{l-k}\alpha=\beta} c_k^{'}(\alpha) \\ C_l^{''}(\beta) &= \sum_{k=k_0}^{l} \mu^{l-k} \sum_{\pi^{l-k}\alpha=\beta} c_k^{''}(\alpha). \end{aligned} \tag{5.4}$$

These coefficients satisfy the *recursion relations*

$$\begin{aligned} C_{k_0}^{'}(\beta) &= c_{k_0}^{'}(\beta) \\ C_l^{'}(\beta) &= \lambda \sum_{\pi\alpha=\beta} C_{l-1}^{'}(\alpha) + c_l^{'}(\beta), \end{aligned} \tag{5.5}$$

and similarly for $C_l^{''}(\beta)$, with μ instead of λ. In terms of the $C_l^{'}(\beta)$ and the $C_l^{''}(\beta)$ we have
(5.6)

$$\phi(x) = \varepsilon(x) \sum_{l=k_0}^{\infty} \sum_{\beta\in W} C_l^{'}(\beta)\psi_\beta(-\pi^{-l}x)\phi_l(x) + \sum_{l=k_0}^{\infty} \sum_{\beta\in W} C_l^{''}(\beta)\psi_\beta(-\pi^{-l}x)\phi_l(x).$$

Invoking the Fourier expansion of $\varepsilon(x)\phi_l(x)$ (see [K-dS12], Corollary 2.2) we finally get the formula

$$C_l(\beta) = \frac{\tau(\varepsilon^{-1})}{q^\nu} \sum_{u\in U_F/U_F^\nu} \varepsilon^{-1}(u) C_l^{'}(\beta - \pi^{-\nu}u) + C_l^{''}(\beta). \tag{5.7}$$

Here U_F^ν denotes the group of units which are congruent to 1 modulo π^ν, and $\tau(\varepsilon^{-1})$ is the Gauss sum

$$\tau(\varepsilon^{-1}) = \sum_{u\in U_F/U_F^\nu} \psi(\pi^{-\nu}u)\varepsilon(u). \tag{5.8}$$

We recall the well-known identity

$$\tau(\varepsilon)\tau(\varepsilon^{-1}) = \varepsilon(-1)q^\nu. \tag{5.9}$$

5.2. **Operators on functions on W.** As in [K-dS12], Section 3.4, we introduce some operators on the space $\mathcal{C}$ of E-valued functions on W with finite support. If $f \in \mathcal{C}$ we define

- The *suspension* of f

$$Sf(\beta) = \sum_{\pi\alpha=\beta} f(\alpha). \tag{5.10}$$

- The *convolution* of f with a character ξ of U_F, of conductor $\nu \geq 1$

$$E_\xi f(\beta) = \frac{\tau(\xi^{-1})}{q^\nu} \sum_{u\in U_F/U_F^\nu} \xi^{-1}(u) f(\beta - \pi^{-\nu}u). \tag{5.11}$$

- The operator Π

$$\Pi f(\beta) = f(\pi\beta). \tag{5.12}$$

We decompose $\mathcal{C}$ as a direct sum $\mathcal{C} = \mathcal{C}_0 \bigoplus \mathcal{C}_1$, where

$$\begin{aligned} \mathcal{C}_0 &= \left\{ f | \forall \beta, \quad \sum_{\pi t = 0} f(\beta + t) = 0 \right\} \\ \mathcal{C}_1 &= \{ f | f(\beta) \text{depends only on } \pi\beta \}. \end{aligned} \tag{5.13}$$

Lemma 5.1. *(i) The projection onto* $\mathcal{C}_1$ *is*

$$P_1 = \frac{1}{q} \Pi S. \tag{5.14}$$

(ii) Let ξ *be any non-trivial character. Then the projection onto* $\mathcal{C}_0$ *is*

$$P_0 = E_\xi E_{\xi^{-1}} = E_{\xi^{-1}} E_\xi. \tag{5.15}$$

(iii) If ξ *is non-trivial then* $SE_\xi = 0$ *and* $E_\xi E_{\xi^{-1}} E_\xi = E_\xi$.

Proof. All the statements are elementary, and best understood if we associate to f its Fourier transform

$$\hat{f}(x) = \sum_{\beta \in W} f(\beta) \psi_\beta(x) \tag{5.16}$$

($x \in \mathcal{O}_F$) and apply Lemma 2.1. See [K-dS12], Section 3.4.

□

For $f, g_1, \ldots, g_r \in \mathcal{C}$ we write $f = O(g_1, \ldots, g_r)$ to mean that in the sup norm $||f|| \leq \max ||g_i||$.

5.3. **Conclusion of the proof in the tamely ramified case.** We assume from now on that $\nu = 1$, i.e. ε is tamely ramified. The Breuil-Schneider estimates on λ and μ are

$$\begin{aligned} |\pi^{-m}| &\leq |\lambda|, |\mu| \leq |q^{-1}\pi^{-m}| \\ |\lambda\mu| &= |q^{-1}\pi^{-2m}|. \end{aligned}$$

Fix a $\phi \in \Lambda$ as in (5.3), so that

$$c_k^{'}, c_k^{''} = O(\pi^{-mk}), \tag{5.17}$$

and assume that it vanishes off $\mathcal{O}_F$. We shall prove by increasing induction on l that for $l \leq 0$

$$C_l^{'},\ C_l^{''} = O(q^{-1}\pi^{-ml}). \tag{5.18}$$

When we reach $l = 0$ this will imply Theorem 1.2, even uniformly in β, thanks to the fact that the algebraic part of ρ is essentially trivial.

Using the notation of the last sub-section, we can write the recursion relations (5.5) as

$$\begin{aligned} C_{k_0}^{'} &= c_{k_0}^{'}, \quad C_{k_0}^{''} = c_{k_0}^{''} \\ C_l^{'} &= \lambda S C_{l-1}^{'} + c_l^{'} \\ C_l^{''} &= \mu S C_{l-1}^{''} + c_l^{''}. \end{aligned} \tag{5.19}$$

Besides $C_l(\beta)$ we introduce $\tilde{C}_l(\beta)$ so that the following formulae hold

$$\begin{aligned} C_l &= E_\varepsilon C_l' + C_l'' \\ \tilde{C}_l &= E_{\varepsilon^{-1}} C_l'' + C_l'. \end{aligned} \tag{5.20}$$

Here the first formula is just (5.7). The second shows that the amplitudes $\tilde{C}_l(\beta)$ are analogously associated with the function $\tilde{\phi}(x) = \varepsilon^{-1}(x)\phi(x)$.

Next, we observe that since $SE_\varepsilon = SE_{\varepsilon^{-1}} = 0$, we can rewrite the recursion relations as

$$\begin{aligned} C_l' &= \lambda S\tilde{C}_{l-1} + c_l' \\ C_l'' &= \mu S C_{l-1} + c_l''. \end{aligned} \tag{5.21}$$

For $l \leq 0$ the functions C_{l-1} and $\tilde{C}_{l-1}$ belong to the subspace that we have called $\mathcal{C}_1$, because ϕ and $\tilde{\phi}$ vanish on $\pi^{l-1}U_F$. This implies the following result.

Lemma 5.2. *For $l \leq 0$,*

$$\begin{aligned} C_l' &= O(\lambda q\tilde{C}_{l-1}, c_l') \\ C_l'' &= O(\mu q C_{l-1}, c_l''). \end{aligned} \tag{5.22}$$

We can now proceed with the induction. When $l = k_0$ (5.17) clearly implies (5.18). Assume that $l \leq 0$ and that (5.18) has been established up to index $l-1$. As $C_{l-2}' = O(q^{-1}\pi^{-m(l-2)})$, and as $C_{l-2}'' = O(q^{-1}\pi^{-m(l-2)}) = O(q^{-2}\tau(\varepsilon^{-1})\pi^{-m(l-2)})$, we obtain from (5.20) and the fact that $\nu = 1$ the estimate

$$C_{l-2} = O(q^{-2}\tau(\varepsilon^{-1})\pi^{-m(l-2)}). \tag{5.23}$$

By the lemma, this gives

$$C_{l-1}'' = O(\mu q^{-1}\tau(\varepsilon^{-1})\pi^{-m(l-2)}, c_{l-1}'') = O(\mu q^{-1}\tau(\varepsilon^{-1})\pi^{-m(l-2)}) \tag{5.24}$$

(the last equality coming from $|\mu q^{-1}\tau(\varepsilon^{-1})| \geq |\pi^{-m}|$). A second application of (5.20), the identity (5.9), and the induction hypothesis for C_{l-1}' (recall $|\mu| \geq |\pi^{-m}|$) yield

$$\tilde{C}_{l-1} = O(\mu q^{-1}\pi^{-m(l-2)}). \tag{5.25}$$

A second application of the lemma finally gives

$$\begin{aligned} C_l' &= O(\lambda\mu\pi^{-m(l-2)}, c_l'') \\ &= O(q^{-1}\pi^{-ml}, c_l'') = O(q^{-1}\pi^{-ml}). \end{aligned} \tag{5.26}$$

Symmetrically, we get the same estimate on C_l''. This completes the proof of (5.18) at level l, and with it, the proof of Theorem 1.2.

References

[B-B-C] L.Berger, C.Breuil, P.Colmez (Eds.): *Représentationsp -adiques de groupes p-adiques,* Astérisque, **319, 330, 331** (2008-2010).

[Br03] C. Breuil: *Sur quelques représentations modulaires et p-adiques de $GL_2(\mathbb{Q}_p)$II*, J. Inst. Math. Jussieu **2** (2003), 1-36.

[Br04] C.Breuil, *Invariant $\mathcal{L}$ et série spéciale p-adique*, Annales Scientifiques E.N.S. **37** (2004), 559-610.

[Br10] C.Breuil: *The emerging p-adic Langlands programme,* Proceedings of the International Congress of Mathematicians, Hyderabad, India, 2010.

[Br-Sch07] C.Breuil, P.Schneider: *First steps towards p-adic Langlands functoriality,* J. Reine Angew. Math. **610** (2007)149-180.

[Bu98] D.Bump: *Automorphic Forms and Representations*,Cambridge University Press, 1998.

[dI12] M.de Ieso, *Analyse p-adique et complétés unitaires universeles pour* $GL_2(F)$, Ph.D. thesis, Orsay,2012.

[dS08] E.de Shalit: *Integral structures in locally algebraic representations,* unpublished notes (2008).

[Hu09] Y.Hu: *Normes invariantes et existence de filtrations admissibles*, J. Reine Angew. Math. **634** (2009), 107-141.

[K-dS12] D.Kazhdan, E.de Shalit: *Kirillov models and integral structures in p-adic smooth representations of* $GL_2(F)$, J.Algebra **353** (2012), 212-223.

[P01] D.Prasad, *appendix* to P.Schneider, J.Teitelbaum: $U(\mathfrak{g})$*-finite locally analytic representations,* Representation Theory **5** (2001), 111-128.

[So13] C.Sorensen: *A proof of the Breuil-Schneider conjecture in the indecomposable case*, Annals of Mathematics **177**(2013),1-16.

[T93] J.Teitelbaum, *Modular representations of* PGL_2 *and automorphic forms for Shimura curves*, Invent. Math. **113**(1993), 561-580.

[V08] M.-F.Vigneras, *A criterion for integral structures and coefficient systems on the tree of* $PGL(2, F)$, Pure and Appl. Math. Quat. **4** (2008), 1291-1316.

3 Existence of Invariant Norms in p-adic Representations of $GL_2(F)$ of Large Weights

Eran Assaf
Unpublished

Existence of Invariant Norms in p-adic Representations of $GL_2(F)$ of Large Weights

Eran Assaf

Abstract

In [4] Breuil and Schneider formulated a conjecture on the equivalence of the existence of invariant norms on certain p-adically locally algebraic representations of $GL_n(F)$ and the existence of certain de-Rham representations of $Gal(\overline{F}/F)$, where F is a finite extension of $\mathbb{Q}_p$. In [3, 9] Breuil and de Ieso proved that in the case $n = 2$ and under some restrictions, the existence of certain admissible filtrations on the ϕ-module associated to the two-dimensional de-Rham representation of $Gal(\overline{F}/F)$ implies the existence of invariant norms on the corresponding locally algebraic representation of $GL_2(F)$. In [3, 9], there is a significant restriction on the weight - it must be small enough. In [5] the conjecture is proved in greater generality, but the weights are still restricted to the extended Fontaine-Laffaille range. In this paper we prove that in the case $n = 2$, even with larger weights, under some restrictions, the existence of certain admissible filtrations implies the existence of invariant norms.

1. Introduction, Notation and Main Results

1.1. Introduction

Let p be a prime number. Let F be a finite extension of $\mathbb{Q}_p$, and let C be a finite extension of $\mathbb{Q}_p$ which is "large enough" in a precise way to be defined in Section 2. This paper lies in the framework of the p-adic local Langlands programme, whose goal is to associate to certain n-dimensional continuous p -adic representations of $Gal(\overline{F}/F)$, certain representations of $G = GL_n(F)$.

If $F = \mathbb{Q}_p$ and $n = 2$, then this is essentially well understood - one has a correspondence $V \mapsto \Pi(V)$ ([6],[13],[8]) associating to a 2-dimensional C-representation V of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, a unitary admissible representation of $GL_2(\mathbb{Q}_p)$. This correspondence is compatible with the classical local Langlands correspondence and with completed étale cohomology ([10]).

Other cases seem somewhat more delicate. In particular, Breuil and Schneider have formulated in [4] a conjecture, generalizing a previous conjecture of Schneider and Teitelbaum [16], which reveals a deep connection between the category of n-dimensional continuous de-Rham representations of $Gal(\overline{F}/F)$, and certain locally algebraic representations of $GL_n(F)$.

Preprint submitted to Elsevier *April 14, 2017*

By the theory of Colmez and Fontaine ([7]), one knows that a de-Rham representation of $Gal(\overline{F}/F)$, V, is equivalent to a vector space, $D = D_{dR}(V)$, equipped with an action of the Weil-Deligne group of F and a filtration, such that the filtration and the action satisfy a certain relation called *weak admissibility*. To this object, called the filtered (ϕ, N)-module attached to V, one can associate a smooth representation π of $GL_n(F)$ by a slight modification of the classical local Langlands correspondence ([4], p. 16-17). On the other hand, the Hodge-Tate weights of the filtration give rise to an irreducible algebraic representation of $GL_n(F)$, which we denote by ρ. The Breuil-Schneider conjecture essentially says that the existence of a weakly admissible filtration on D must be equivalent to the existence of a $GL_n(F)$-invariant norm on the locally algebraic representation $\rho \otimes \pi$. We mention that partial results, in this generality, have been obtained by Hu ([12]), who proved that the existence of an invariant norm on $\rho \otimes \pi$ implies the existence of a weakly admissinble filtration on D, and Sorensen ([18]), who proved the equivalence when π is essentially discrete series.

In this paper we consider the particular case where $n = 2$, and the representation of the Galois group is crystalline.

Let D be a ϕ-module of rank 2 over $F \otimes_{\mathbb{Q}_p} C$, equipped with a weakly admissible filtration. Imposing some additional technical restrictions on the weights of the filtration and on the smooth part, we show in this paper that the locally algebraic representation $\Pi(D)$ associated to D according to the above process admits a G-invariant norm. The methods we employ in order to prove this result are well-known and were previously employed by Breuil ([3]) and de Ieso ([9]). The novelty of this paper is the extension of these methods to larger weights, even though this is accompanied by a substantial restriction on the smooth representation, π.

We remark that in [5], the authors have proved many cases of the conjecture formulated by Breuil and Schneider, using global methods. However, the results we obtain in this paper are not included in their work, as they restrict the weights to be in the extended Fontaine-Laffaille range, which, for $n = 2$, means that the weight is small.

1.2. Notation

Let p be a prime number. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and a finite extension F of $\mathbb{Q}_p$, contained in $\overline{\mathbb{Q}}_p$. Denote by $\mathcal{O}_F$ the ring of integers of F, by $\mathfrak{p}_F$ its maximal ideal, and by $\kappa_F = \mathcal{O}_F/\mathfrak{p}_F$ its residue field. We also fix a uniformizer $\varpi = \varpi_F \in \mathfrak{p}_F$.

Denote by C a finite extension of $\mathbb{Q}_p$ satisfying $|S| = [F : \mathbb{Q}_p]$, where $S := Hom_{alg}(F, C)$, and containing a square root of $\sigma(\varpi)$ for every $\sigma \in S$.

Denote by $\mathcal{O}_C$ the ring of integers of C, by $\mathfrak{p}_C$ its maximal ideal, and by $\kappa_C = \mathcal{O}_C/\mathfrak{p}_C$ its residue field. We also fix a uniformizer $\varpi = \varpi_C \in \mathfrak{p}_C$.

We denote $f = [\kappa_F : \mathbb{F}_p]$, $q = p^f$ the size of the residue field, and by e we denote the ramification index of F over $\mathbb{Q}_p$, so that $[F : \mathbb{Q}_p] = ef$ and $\kappa_F \simeq \mathbb{F}_q$. We denote by $F_0 = \mathrm{Frac}(W(\kappa_F))$ the maximal unramified subfield of F, and by φ_0

the absolute Frobenius of degree p in $Gal(F_0/\mathbb{Q}_p)$. We denote by $Gal(\overline{F}/F)$ the Galois group of F and by $W(\overline{F}/F)$ its Weil group. Class field theory gives rise to a homomorphism $\mathrm{rec} : W(\overline{F}/F)^{ab} \to F^\times$ (Artin reciprocity map) which we normalize by sending the coset of the arithmetic Frobenius to $\varpi^{-1}\mathcal{O}_F^\times$.

Denote by $v = v_F$ the p-adic valuation on $\overline{\mathbb{Q}}_p$ normalized by $v_F(\varpi) = 1$. If $x \in \overline{F}$, we let $|x| = q^{-v_F(x)}$. If $\lambda \in \kappa_F$, we denote by $[\lambda]$ the Teichmüller representative of λ in $\mathcal{O}_F$. If $\mu \in C^\times$, we denote by $\mathrm{nr}(\mu) : F^\times \to C^\times$ the unramified character sending ϖ to μ.

Denote by $\mathbf{G}$ the algebraic group $\mathbf{GL}_2$ defined over $\mathcal{O}_F$, and let $G = \mathbf{G}(F)$ be its F-points.

Let $\mathbf{B}$ be the Borel subgroup of $\mathbf{G}$ consisting of upper triangular matrices, and let $B = \mathbf{B}(F)$ be its F-points.

Let $\mathbf{N}$ be the unipotent radical of $\mathbf{B}$, and let $N = \mathbf{N}(F)$ be its F-points.

Let K be the group $GL_2(\mathcal{O}_F)$, which is, up to conjugation, the unique maximal compact subgroup of G. Let I be the Iwahori subgroup of K corresponding to B, and let $I(1)$ be its pro-p-Iwahori.

Recall that the reduction mod $\mathfrak{p}_F$ induces a surjective homomorphism

$$\mathrm{red} : K \to \mathbf{G}(\kappa_F)$$

and that $I = \mathrm{red}^{-1}(\mathbf{B}(\kappa_F))$ and $I(1) = \mathrm{red}^{-1}(\mathbf{N}(\kappa_F))$.

We denote by $Z \simeq F^\times$ the center of G, and denote

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \alpha w = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}.$$

If $\lambda \in \mathcal{O}_F$, we denote

$$w_\lambda = \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}.$$

If $\underline{n} = (n_\sigma)_{\sigma \in S}, \underline{m} = (m_\sigma)_{\sigma \in S}$ are elements of $\mathbb{Z}_{\geq 0}^S$, we write:

(i) $\underline{n}! = \prod_{\sigma \in S} n_\sigma!$

(ii) $|\underline{n}| = \sum_{\sigma \in S} n_\sigma$

(iii) $\underline{n} - \underline{m} = (n_\sigma - m_\sigma)_{\sigma \in S}$

(iv) $\underline{n} \leq \underline{m}$ if $n_\sigma \leq m_\sigma$ for all $\sigma \in S$

(v) $\binom{\underline{n}}{\underline{m}} = \frac{\underline{n}!}{\underline{m}!(\underline{n}-\underline{m})!}$

(vi) If $z \in \mathcal{O}_F$, we write $z^{\underline{n}} = \prod_{\sigma \in S} \sigma(z)^{n_\sigma}$.

1.3. Main Results

We fix $(\lambda_1, \lambda_2) \in C^\times \times C^\times$ such that $\lambda_1\lambda_2^{-1} \notin \{q^2, 1\}$ and $\underline{k} \in \mathbb{Z}_{\geq 0}^S$. Denote

$$S^+ = \{\sigma \in S \mid k_\sigma \neq 0\} \subseteq S$$

We also fix some $\iota \in S$, and partition S^+ according to the action of $\sigma \in S^+$ on the residue field. More precisely, for each $l \in \{0, \ldots, f-1\}$, denote

$$J_l = \{\sigma \in S^+ \mid \sigma([\zeta]) = \iota \circ \varphi_0^l([\zeta]) \quad \forall \zeta \in \kappa_F\}.$$

For example, if F is unramified, then $|J_l| \leq 1$ for all l.
If $i \in \mathbb{Z}$, we denote by $\bar{i}$ the unique representative of $i \mod f$ in $\{0, \ldots, f-1\}$.
For $\sigma \in J_l$, we denote

$$v_\sigma = \inf \left\{ i \mid 1 \leq i \leq f, \quad J_{\overline{l+i}} \neq \emptyset \right\}$$

that is, the smallest power of Frobenius φ_0 that is needed to pass from J_l to another, nonempty J_k.
We denote by $\chi : GL_2(F) \to F^\times$ the character defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \varpi^{-v_F(ad-bc)}$$

For $k \in \mathbb{Z}_{\geq 0}$, we denote by ρ_k the irreducible algebraic representation of $\mathbf{G}$ of highest weight $\mathrm{diag}(x_1, x_2) \mapsto x_2^k$ with respect to $\mathbf{B}$, the Borel subgroup of upper triangular matrices.
We regard it also as a representation of $G = \mathbf{G}(F)$, and for any $\sigma \in S$, denote by ρ_k^σ the base change of ρ_k to a representation of $G \otimes_{F,\sigma} C$.
Also, for any $\sigma \in S$, we fix a square root of $\sigma(\varpi)$ and write $\underline{\rho}_k^\sigma = \rho_k^\sigma \otimes_C (\sigma \circ \chi)^{\frac{k}{2}}$.
For $\underline{k} \in \mathbb{Z}_{\geq 0}^S$, we write

$$\rho_{\underline{k}} = \bigotimes_{\sigma \in S} \rho_{k_\sigma}^\sigma, \quad \underline{\rho}_{\underline{k}} = \bigotimes_{\sigma \in S} \underline{\rho}_{k_\sigma}^\sigma$$

Let $\mathbf{T}$ be the standard maximal torus of $\mathbf{B}$ consisting of diagonal matrices, and let $T = \mathbf{T}(F)$.

Definition 1.1. Let $\theta : T \to C^\times$ be a C-character of T inflated to B, via $T \simeq B/N$. The smooth *principal series representation* corresponding to θ is

$$Ind_B^G(\theta) = \left\{ f : G \to C \mid \begin{array}{c} \exists U_f \text{ open s.t. } f(bgk) = \theta(b) f(g) \\ \forall g \in G, \quad b \in B, \quad k \in U_f \end{array} \right\}$$

with the group G acting by right translations, namely $(gf)(x) = f(xg)$ for all $x, g \in G$ and $f \in Ind_B^G(\theta)$.

Finally, we denote by

$$\pi = Ind_B^G(\mathrm{nr}(\lambda_1^{-1}) \otimes \mathrm{nr}(\lambda_2^{-1}))$$

the smooth unramified parabolic induction.
Note that the hypothesis on (λ_1, λ_2) assures us that π is irreducible.

We shall from now on consider the irreducible locally algebraic representations of the form $\rho_{\underline{k}} \otimes \pi$.

Note that ρ_k is not the most general irreducible algebraic representation of $\mathbf{G}$, as it can be twisted by a power of the determinant.

However, for the purpose of existence of G-invariant norms, a twist by a power of the determinant is equivalent to a twist by a power of χ, which can be then absorbed by π into the values of λ_1, λ_2.

The Breuil-Schneider conjecture can be reformulated as follows (see [9])

Conjecture 1.2. *The following two statements are equivalent:*

(i) The representation $\rho_{\underline{k}} \otimes \pi$ admits a G-invariant norm, i.e. a p-adic norm such that $\|gv\| = \|v\|$ for all $g \in G$ and $v \in \rho_{\underline{k}} \otimes \pi$.

(ii) The following inequalities are satisfied:

- $v_F(\lambda_1^{-1}) + v_F(\lambda_2^{-1}) + |\underline{k}| = 0$
- $v_F(\lambda_2^{-1}) + |\underline{k}| \geq 0$
- $v_F(q\lambda_1^{-1}) + |\underline{k}| \geq 0$

The implication $(i) \Rightarrow (ii)$ of Conjecture 1.2 follows from the work of Hu, which shows it in full generality (for $GL_n(F)$) in [12], using a result of Emerton ([11], Lemma 1.6).

It remains to show $(ii) \Rightarrow (i)$.

The case $\lambda_1 \in \mathcal{O}_C^\times$ (resp. $q\lambda_2 \in \mathcal{O}_C^\times$) is treated in [9, Prop. 4.10] hence we may assume that $\lambda_1, q\lambda_2 \notin \mathcal{O}_C^\times$.

In [3, 9] Breuil and de Ieso represent $\rho_{\underline{k}} \otimes \pi$ as a quotient of a compact induction.

We briefly recall the definition of locally algebraic compact induction.

Definition 1.3. Let G be a topological group, and let H be a closed subgroup. Let R be either $\mathcal{O}_C$ or C. Let (π, V) be an R-linear representation of H over a free R-module of finite rank V. We denote by $ind_H^G \pi$ or by $ind_H^G V$ the locally algebraic compact induction of (π, V) from H to G. The space of the representation is

$$ind_H^G \pi = \left\{ f : G \to V \mid \begin{array}{c} f(hg) = \pi(h) f(g) \quad \forall h \in H \\ f \text{ has compact support mod } H, \quad f \text{ is locally algebraic} \end{array} \right\}$$

and G acts on $ind_H^G \pi$ by right translation, i.e. $(gf)(x) = f(xg)$ for all $g, x \in G$.

Then

$$\rho_{\underline{k}} \otimes \pi \simeq \frac{ind_{KZ}^G \rho_{\underline{k}}}{(T-a) ind_{KZ}^G \rho_{\underline{k}}} =: \Pi_{\underline{k},a}$$

where $a = \lambda_1 + q\lambda_2 \in \mathfrak{p}_C$, $\rho_{\underline{k}}^0$ is an $\mathcal{O}_C$-lattice in $\rho_{\underline{k}}$, ind_{KZ}^G denotes the compact induction, and T is the usual Hecke operator [1].

We then have a natural map

$$\theta : \frac{ind_{KZ}^{G}\rho_{\underline{k}}^{0}}{(T-a)(ind_{KZ}^{G}\rho_{\underline{k}}^{0})} \to \Pi_{\underline{k},a}$$

whose image is denoted by $\Theta_{\underline{k},a}$.

This is a sub-$\mathcal{O}_C[K]$-module of finite type which generates $\rho_{\underline{k}} \otimes \pi$ over C.

Proving Conjecture 1.2 is then equivalent to proving that $\Theta_{\underline{k},a}$ is separated, i.e. does not contain a C-line (see [11, Prop. 1.17]) . In this paper, we prove that this is the case, for some additional values of $\underline{k}$ and a.

This generalizes the previous works of Breuil and de Ieso in [3, 9], using similar methods.

In fact, de Ieso proves the following theorem:

Theorem 1.4. *We follow the preceding notations. The morphism θ is injective if and only if the following two conditions are satisfied:*

(i) For all $l \in \{0, \dots, f-1\}$, $\quad |J_l| \leq 1$.

(ii) For all $\sigma \in J_l$

$$k_\sigma + 1 \leq p^{v_\sigma}.$$

As a corollary, it follows that under these conditions $\Theta_{\underline{k},a}$ is separated.

In this paper, we prove that even in some cases where θ is not injective, the lattice $\Theta_{\underline{k},a}$ is still separated. Namely, we prove the following theorem:

Theorem 1.5. *We follow the preceding notations. Assume that $|S^+| = 1$, denote by σ the unique element in S^+, and let $k = k_\sigma = d \cdot q + r$, with $0 \leq r < q$. Assume that one of the following three conditions is satisfied:*

(i) $k \leq \frac{1}{2}q^2$ with $r < q - d$ and $v_F(a) \in [0, 1]$.

(ii) $k \leq \frac{1}{2}q^2$ with $2v_F(a) - 1 \leq r < q - d$ and $v_F(a) \in [1, e]$.

(ii) $k \leq \min\left(p \cdot q - 1, \frac{1}{2}q^2\right)$, $d - 1 \leq r$ and $v_F(a) \geq d$.

Then $\Theta_{\underline{k},a}$ is separated.

Therefore, these conditions on $\underline{k}, a$ ensure the existence of a G-invariant norm on $\rho_{\underline{k}} \otimes \pi$, establishing new cases of Conjecture 1.2.

Example 1.6. Here are a couple of explicit examples for the established new cases:

1. Let $p \neq 2$, $k = \frac{1}{2}(q^2 - 1)$ and $v_F(a) \in [0, \min(e, \frac{q+1}{4})]$. Then, as $k = \frac{1}{2}(q-1)q + \frac{1}{2}(q-1)$, we see that $d = r = \frac{1}{2}(q-1)$, hence

$$2v_F(a) - 1 \leq 2 \cdot \frac{q+1}{4} - 1 = \frac{1}{2}(q-1) = r < q - d = \frac{1}{2}(q+1)$$

   so either (i) or (ii) in Theorem 1.5 is satisfied, showing that the lattice $\Theta_{\underline{k},a}$ is separated in this case.

2. Let $q = p \neq 2$, $k = \frac{1}{2}(p^2-1)$ and $v_F(a) \geq \frac{1}{2}(p-1)$. As in the previous example, $d = r = \frac{1}{2}(p-1)$, hence $d-1 \leq r$, and $v_F(a) \geq d$. This shows that condition *(iii)* in Theorem 1.5 is satisfied, showing that the lattice $\Theta_{\underline{k},a}$ is separated in this case.

2. Preliminaries

2.1. The Bruhat-Tits Tree

We refer to [2] and [17] for further details concerning the construction and properties of the Bruhat-Tits tree of G.

Let $\mathcal{T}$ be the Bruhat-Tits tree of G: its vertices are in equivariant bijection with the left cosets G/KZ.

The tree $\mathcal{T}$ is equipped with a combinatorial distance, and G acts on it by isometries.

We denote by s_0 the *standard vertex*, corresponding to the trivial class KZ.

Equivalently, as the vertices are in equivariant bijection with homothety classes of lattices in F^2, s_0 corresponds to the homothety class of the lattice $\mathcal{O}_F \oplus \mathcal{O}_F$.

For $n \geq 0$, we call the collection of vertices in $\mathcal{T}$ at distance n from the standard vertex s_0, *the circle of radius* n.

Recall that we have the Cartan decomposition

$$G = \coprod_{n\in\mathbb{N}} KZ\alpha^{-n}KZ = \left(\coprod_{n\in\mathbb{N}} IZ\alpha^{-n}KZ\right)\coprod\left(\coprod_{n\in\mathbb{N}} IZ\beta\alpha^{-n}KZ\right). \quad (1)$$

In particular, for any $n \in \mathbb{N}$, the classes of $KZ\alpha^{-n}KZ/KZ$ correspond to vertices s_i of $\mathcal{T}$ such that $d(s_i, s_0) = n$. Denote $I_0 = \{0\}$, and for any $n \in \mathbb{N}_{>0}$

$$I_n = \left\{[\mu_0] + \varpi[\mu_1] + \ldots + \varpi^{n-1}[\mu_{n-1}] \mid (\mu_0, \ldots, \mu_{n-1}) \in \kappa_F^n\right\} \subseteq \mathcal{O}_F$$

is a set of representatives for $\mathcal{O}_F/\varpi^n\mathcal{O}_F$.

For $n \in \mathbb{N}$ and $\mu \in I_n$, we denote :

$$g^0_{n,\mu} = \begin{pmatrix} \varpi^n & \mu \\ 0 & 1 \end{pmatrix}, \quad g^1_{n.\mu} = \begin{pmatrix} 1 & 0 \\ \varpi\mu & \varpi^{n+1} \end{pmatrix}.$$

We note that $g^0_{0,0}$ is the identity matrix, $g^1_{0,0} = \alpha$ and that, for all $n \in \mathbb{N}$ and any $\mu \in I_n$, we have $g^1_{n,\mu} = \beta g^0_{n,\mu} w$. Then, $g^0_{n,\mu}$ and $g^1_{n,\mu}$ define a system of representatives for G/KZ:

$$G = \left(\coprod_{n\in\mathbb{N},\mu\in I_n} g^0_{n,\mu}KZ\right)\coprod\left(\coprod_{n\in\mathbb{N},\mu\in I_n} g^1_{n,\mu}KZ\right). \quad (2)$$

For $n \in \mathbb{N}$ we denote

$$S_n^0 = IZ\alpha^{-n}KZ = \coprod_{\mu \in I_n} g_{n,\mu}^0 KZ, \quad S_n^1 = IZ\beta\alpha^{-n}KZ = \coprod_{\mu \in I_n} g_{n,\mu}^1 KZ$$

and we let $S_n = S_n^0 \coprod S_n^1$ and $B_n = B_n^0 \coprod B_n^1$, where $B_n^0 = \coprod_{m \leq n} S_m^0$ and $B_n^1 = \coprod_{m \leq n} S_m^1$.

In particular, we have $S_0 = KZ \coprod \alpha KZ$.

Remark 2.1. Recall, as in [2, 9] that $S_n^0 \coprod S_{n-1}^1$ (resp. $B_n^0 \coprod B_{n-1}^1$) is the collection of vertices in $\mathcal{T}$ at distance n (resp. at most n) from s_0. Similarly, $S_n^1 \coprod S_{n-1}^0$ (resp. $B_n^1 \coprod B_{n-1}^0$) is the collection of vertices in $\mathcal{T}$ at distance n (resp. at most n) from αs_0.

We denote by R either the field C or its ring of integers $\mathcal{O}_C$. Let σ be a continuous R-linear representation of KZ on a free R-module of finite rank V_σ. We denote by $ind_{KZ}^G \sigma$ the R-module of functions $f : G \to V_\sigma$ compactly supported modulo Z, such that

$$f(\kappa g) = \sigma(\kappa) f(g) \quad \forall \kappa \in KZ, g \in G$$

with G acting by right translations, i.e. $(g \cdot f)(g') = f(g'g)$.

As in [1], for $g \in G$, $v \in V_\sigma$, we denote by $[g, v]$ the element of $ind_{KZ}^G \sigma$ supported on KZg^{-1} and such that $[g, v](g^{-1}) = v$.

Then we have

$$\forall g, g' \in G, v \in V_\sigma \quad g \cdot [g', v] = [gg', v], \quad \forall g \in G, \kappa \in KZ, v \in V_\sigma \quad [g\kappa, v] = [g, \sigma(\kappa) v]$$

We can think of $ind_{KZ}^G \sigma$ as a vertex coefficient system on $\mathcal{T}$, having σ as the module on each vertex, identifying $[g, v]$ with the vector v at the vertex corresponding to g, i.e. identifying vertex g with KZg^{-1}. Note that the choice of representative for gKZ affects the choice of vector $v \in \sigma$.

Recall the following result ([1, §2]), which gives a basis for the $R[G]$-module $ind_{KZ}^G \sigma$.

Proposition 2.2. *Let $\mathcal{B}$ be a basis for V_σ over R, and let $\mathcal{G}$ be a system of representatives for left cosets of G/KZ. Then the family of functions $\mathcal{I} := \{[g, v] \mid g \in \mathcal{G}, v \in \mathcal{B}\}$ forms a basis for $ind_{KZ}^G \sigma$ over R.*

Remark 2.3. The representation $ind_{KZ}^G \sigma$ is isomorphic to the representation of G given by the $R[G]$-module $R[G] \otimes_{R[KZ]} V_\sigma$. More precisely, if $g \in G$ and $v \in V_\sigma$, then the element $g \otimes v$ corresponds to the function $[g, v]$.

From proposition 2.2 and the decomposition (2), any function $f \in ind_{KZ}^G \sigma$ can be written uniquely as a finite sum of the form

$$f = \sum_{n=0}^{n_0} \sum_{\mu \in I_n} \left([g_{n,\mu}^0, v_{n,\mu}^0] + [g_{n,\mu}^1, v_{n,\mu}^1] \right)$$

with $v^0_{n.\mu}, v^1_{n,\mu} \in V_\sigma$, and where n_0 is a non-negative integer, which depends on f. We call the *support* of f the collection of $g^i_{n,\mu}$ such that $v^i_{n,\mu} \neq 0$. We write $f \in S_n$ (resp. B_n, S^0_n, etc.) if the support of f is contained in S_n (resp. B_n, S^0_n, etc.). We write $f \in B^0$ if the support of f is contained in B^0_n for some n, and $f \in B^1$ if the support of f is contained in B^1_n for some n.

Let π be a continuous R-linear representation of G over an R-module. From [1], we have a canonical isomorphism of R-modules

$$Hom_{R[G]}(ind^G_{KZ}\sigma, \pi) \simeq Hom_{R[KZ]}(\sigma, \pi\mid_{KZ})$$

which translates to the fact that the functor of compact induction ind^G_{KZ} is left adjoint to the restriction functor, and is called *compact Frobenius reciprocity.*

2.2. Hecke Algebras

Let σ be a continuous R-linear representation of KZ over a free R-module V_σ of finite rank. The Hecke algebra $\mathcal{H}(KZ, \sigma)$ associated to KZ and σ is the R-algebra defined by

$$\mathcal{H}(KZ, \sigma) = End_{R[G]}(ind^G_{KZ}\sigma).$$

We can interpret $\mathcal{H}(KZ, \sigma)$ as a convolution algebra. In fact, denote by $\mathcal{H}_{KZ}(\sigma)$ the R-module of functions $\varphi : G \to End_R(V_\sigma)$ compactly supported modulo Z, such that

$$\forall \kappa_1, \kappa_2 \in KZ, \quad \forall g \in G, \quad \varphi(\kappa_1 g \kappa_2) = \sigma(\kappa_1) \circ \varphi(g) \circ \sigma(\kappa_2).$$

This is a unitary R-algebra with the convolution product defined, for all $\varphi_1, \varphi_2 \in \mathcal{H}_{KZ}(\sigma)$ and all $g \in G$, by the following formula:

$$\varphi_1 * \varphi_2(g) = \sum_{xKZ \in G/KZ} \varphi_1(x) \circ \varphi_2(x^{-1}g).$$

It admits as a unit element the function $\varphi_e = [1, id]$ defined by

$$\varphi_e(g) = \begin{cases} \sigma(g) & g \in KZ \\ 0 & else \end{cases}.$$

One may verify that the bilinear map

$$\begin{array}{rcl} \mathcal{H}_{KZ}(\sigma) \times ind^G_{KZ}\sigma & \to & ind^G_{KZ}\sigma \\ (\varphi, f) & \mapsto & T_\varphi(f)(g) := \displaystyle\sum_{xKZ \in G/KZ} \varphi(x)\left(f(x^{-1}g)\right) \end{array}$$

equips $ind^G_{KZ}\sigma$ with the structure of a left $\mathcal{H}_{KZ}(\sigma)$-module, which commutes with the action of G.

The following Lemma is well known, see e.g. [9, Lemma 2.4].

Lemma 2.4. *The map*

$$\begin{array}{rcl} \mathcal{H}_{KZ}(\sigma) & \to & \mathcal{H}(KZ,\sigma) \\ \varphi & \mapsto & T_\varphi(f) \end{array}$$

is an isomorphism of R-algebras. In particular, if $g \in G$, and if $v \in V_\sigma$, the action of T_φ on $[g,v]$ is given by

$$T_\varphi([g,v]) = \sum_{xKZ \in G/KZ} [gx, \varphi(x^{-1})(v)]. \tag{3}$$

We assume now that $R = C$. Denote by $\mathbf{1}$ the trivial representation of KZ and assume that σ is the restriction to KZ of a locally analytic representation (in the sense of [15, 14]) of G on V_σ. By [16], the map

$$\begin{array}{rcl} \iota_\sigma : \mathcal{H}_{KZ}(\mathbf{1}) & \to & \mathcal{H}_{KZ}(\sigma) \\ \varphi & \mapsto & (\varphi \cdot \sigma)(g) := \varphi(g)\sigma(g) \end{array}$$

is then an injective homomorphism of C-algebras. Before we state a condition assuring the bijectivity of ι_σ, we recall the existence of a $\mathbb{Q}_p$-linear action of the Lie algebra $\mathfrak{g}$ of G on the space V_σ defined by

$$\forall \mathfrak{x} \in \mathfrak{g}, \forall v \in V_\sigma, \quad \mathfrak{x}v = \frac{d}{dt}\exp(t\mathfrak{x})v \mid_{t=0}$$

where $\exp : \mathfrak{g} \dashrightarrow G$ denotes the exponential map defined locally in the neighbourhood of 0 ([[14, §2]).

This action is extended to an action of the Lie algebra $\mathfrak{g} \otimes_{\mathbb{Q}_p} C$, and allows de Ieso to obtain the following result: (see [9, Lemma 4.2.5])

Lemma 2.5. *If the $\mathfrak{g} \otimes_{\mathbb{Q}_p} C$-module V_σ is absolutely irreducible, then the map ι_σ is bijective.*

3. Representations of $GL_2(F)$

3.1. $\mathbb{Q}_p$-algebraic representations of $GL_2(F)$

For $k \in \mathbb{N}$, we denote by ρ_k the irreducible algebraic representation of $\mathbf{G}$ of highest weight $\mathrm{diag}(x_1, x_2) \mapsto x_2^k$ with respect to $\mathbf{B}$, and we consider it also as a representation of $G = \mathbf{G}(F)$.

For $\sigma \in S$, we denote by ρ_k^σ the base change of ρ_k to a representation of $G \otimes_{F,\sigma} C$.

We denote by $\chi : GL_2(F) \to F^\times$ the character defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \varpi^{-v_F(ad-bc)}.$$

Also, choose a square root of $\sigma(\pi)$ in C, and let

$$\underline{\rho}_k^\sigma = \rho_k^\sigma \otimes_C (\sigma \circ \chi)^{\frac{k}{2}}.$$

For $\sigma \in S$ and $k \in \mathbb{N}$, we identify $\underline{\rho}_k^\sigma$ with the representation of G given by the C-vector space

$$\bigoplus_{i=0}^{k} C \cdot x_\sigma^{k-i} y_\sigma^i$$

of homogeneous polynomials of degree k in x_σ, y_σ with coefficients in C, on which G acts by the following formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x_\sigma^{k-i} y_\sigma^i) = \left(\sigma \circ \chi \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{\frac{k}{2}} (\sigma(a)x_\sigma + \sigma(c)y_\sigma)^{k-i} (\sigma(b)x_\sigma + \sigma(d)y_\sigma)^i. \tag{4}$$

If $w_\sigma \in \underline{\rho}_k^\sigma$ and if $g \in G$, we denote simply gw_σ for the vector obtained from letting g act on w_σ.

Remark 3.1. The formula (4) assures, in particular, that for every $w_\sigma \in \underline{\rho}_k^\sigma$

$$\begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix} w_\sigma = w_\sigma.$$

Fix $\underline{k} = (k_\sigma)_{\sigma \in S} \in \mathbb{N}^S$, and let

$$I_{\underline{k}} = \left\{ \underline{i} = (i_\sigma)_{\sigma \in S} \in \mathbb{N}^S, \quad 0 \le i_\sigma \le k_\sigma \quad \forall \sigma \in S \right\}.$$

We denote by $\rho_{\underline{k}}$ (resp. $\underline{\rho}_{\underline{k}}$) the representation of G on the following vector space

$$V_{\rho_{\underline{k}}} := \bigotimes_{\sigma \in S} \rho_{k_\sigma}^\sigma \quad \left(\text{resp.} \quad V_{\underline{\rho}_{\underline{k}}} := \bigotimes_{\sigma \in S} \underline{\rho}_{k_\sigma}^\sigma\right)$$

on which an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ acts componentwise. In particular, for all $\bigotimes_{\sigma \in S} w_\sigma \in V_{\underline{\rho}_{\underline{k}}}$ we have:

$$\underline{\rho}_{\underline{k}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\bigotimes_{\sigma \in S} w_\sigma \right) = \bigotimes_{\sigma \in S} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} w_\sigma \right).$$

These are two absolutely irreducible representations of G which remain absolutely irreducible even when we restrict them to the action of an open subgroup of G ([4, §2]).

For all $\underline{i} \in I_{\underline{k}}$, we let:

$$e_{\underline{k},\underline{i}} := \bigotimes_{\sigma \in S} e_{k_\sigma, i_\sigma}$$

where, for any $\sigma \in S$, e_{k_σ,i_σ} denotes the monomial $x_\sigma^{k_\sigma - i_\sigma} y_\sigma^{i_\sigma}$. We then denote by $U_{\underline{k}}$ the endomorphism of $V_{\underline{\rho}_{\underline{k}}}$ defined by

$$U_{\underline{k}} := \bigotimes_{\sigma \in S} U_{k_\sigma}^\sigma$$

where U_k^σ denotes, for all $\sigma \in S$ and $k \in \mathbb{N}$, the endomorphism of $\underline{\rho}_k^\sigma$ given, with respect to the basis $(e_{k,i})_{i=0}^k$ by the diagonal matrix

$$U_k^\sigma = \begin{pmatrix} \sigma(\varpi)^k & 0 & \cdots & 0 \\ 0 & \sigma(\varpi)^{k-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

In [9, Lemma 3.2], de Ieso proves the following Lemma.

Lemma 3.2. *There exists a unique function $\psi : G \to End_C(V_{\underline{\rho}_{\underline{k}}})$ supported in $KZ\alpha^{-1}KZ$ such that:*
(i) For any $\kappa_1, \kappa_2 \in KZ$ we have $\psi(\kappa_1 \alpha^{-1} \kappa_2) = \underline{\rho}_{\underline{k}}(\kappa_1) \circ \psi(\alpha^{-1}) \circ \underline{\rho}_{\underline{k}}(\kappa_2)$.
(ii) $\psi(\alpha^{-1}) = U_{\underline{k}}$.

We remark that in fact, $\psi = \underline{\rho}_{\underline{k}} |_{KZ\alpha^{-1}KZ}$, since

$$U_{\underline{k}} = \underline{\rho}_{\underline{k}}(\alpha^{-1}) \tag{5}$$

By Lemma 2.4, we know that the Hecke algebra $\mathcal{H}(KZ, \underline{\rho}_{\underline{k}})$ is naturally isomorphic to the convolution algebra $\mathcal{H}_{KZ}(\underline{\rho}_{\underline{k}})$ of functions $\varphi : G \to End_C(V_{\underline{\rho}_{\underline{k}}})$ compactly supported modulo Z, such that

$$\forall \kappa_1, \kappa_2 \in KZ, g \in G, \quad \varphi(\kappa_1 g \kappa_2) = \underline{\rho}_{\underline{k}}(\kappa_1) \circ \varphi(g) \circ \underline{\rho}_{\underline{k}}(\kappa_2).$$

It follows that the map ψ from Lemma 3.2 corresponds to an operator $T \in \mathcal{H}(KZ, \underline{\rho}_{\underline{k}})$ whose action on the elements $[g, v]$ for $g \in G$ and $v \in V_{\underline{\rho}_{\underline{k}}}$ is given by the formula (3).
Moreover,
Remark 3.3. A simple argument using the Bruhat-Tits tree of G shows that T is injective on $ind_{KZ}^G \underline{\rho}_{\underline{k}}$.

3.2. Lattices

We keep the notations of Section 3.1 and denote by $\underline{\rho}_k^{\sigma,0}$, for $\sigma \in S$ and $k \in \mathbb{N}$, the representation of the group KZ on the $\mathcal{O}_C$-module

$$\bigoplus_{i=0}^{k} \mathcal{O}_C \cdot x_\sigma^{k-i} y_\sigma^i$$

of homogeneous polynomials of degree k, on which an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x_\sigma^{k-i} y_\sigma^i) = (\sigma(a)x_\sigma + \sigma(c)y_\sigma)^{k-i} (\sigma(b)x_\sigma + \sigma(d)y_\sigma)^i$$

and the matrix $\begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix} \in Z$ acts as the identity. If $w_\sigma \in \underline{\rho}_k^{\sigma,0}$ and if $g \in G$, we simply denote by gw_σ the vector obtained from letting g act on w_σ.

Definition 3.4. Let V be a C-vector space. A *lattice* $\mathcal{L}$ in V is a sub-$\mathcal{O}_C$-module of V, such that, for any $v \in V$, there exists a nonzero element $a \in C^\times$ such that $av \in \mathcal{L}$. A lattice $\mathcal{L}$ is called *separated* if $\bigcap_{n\in\mathbb{N}} \varpi^n \mathcal{L} = 0$, which is equivalent to demanding that it contains no C-line.

Example 3.5. The $\mathcal{O}_C$-module $\underline{\rho}_k^{\sigma,0}$ is a separated lattice of $\underline{\rho}_k^{\sigma}$, which is moreover stable under the action of KZ.

Remark 3.6. There are many choices of possible separated lattices in $\underline{\rho}_k^{\sigma}$, which are stable under the action of KZ. Another natural choice (and in some sense even more natural than ours), as pointed out by one of the referees, is the lattice

$$\bigoplus_{i=0}^{k} \mathcal{O}_C \cdot \frac{x_\sigma^{k-i} y_\sigma^i}{(k-i)! \cdot i!}$$

which, in the case $q > p$, is different from $\underline{\rho}_k^{\sigma,0}$. However, as using this lattice facilitates some of the technical aspects, others become more difficult. In particular, we strongly use the divisibilty by powers of p of certain binomial coefficients, which is not possible when using this alternative lattice. Therefore, we have not been able to use different lattices in order to prove more cases of the conjecture. We have further hypothesized the possibility of using different lattices for different values of $v_F(a)$, but this as well did not yield any results.

Example 3.7. We denote by $\underline{\rho}_{\underline{k}}^0$ the representation of KZ on the following space

$$V_{\underline{\rho}_{\underline{k}}^0} = \bigotimes_{\sigma \in S} \underline{\rho}_{k_\sigma}^{\sigma,0}$$

on which an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in KZ$ acts via

$$\underline{\rho}_{\underline{k}}^0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\bigotimes_{\sigma\in S} w_\sigma \right) = \bigotimes_{\sigma\in S} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} w_\sigma \right) \tag{6}$$

The example 3.5 assures us that the $\mathcal{O}_C$-module $V_{\underline{\rho}_{\underline{k}}^0}$ is a separated lattice of the space $V_{\underline{\rho}_{\underline{k}}}$ constructed in Section 3.1. Therefore, the $\mathcal{O}_C$-module $ind_{KZ}^G \underline{\rho}_{\underline{k}}^0$

is also a separated lattice of $ind_{KZ}^{G}\rho_{\underline{k}}$ and is, by construction, stable under the action of G.

By Remark 2.3, we can deduce the existence of an injective map $\mathcal{H}(KZ, \rho_{\underline{k}}^0) \to \mathcal{H}(KZ, \rho_{\underline{k}})$. Moreover, one verifies that the operator $T \in \mathcal{H}(KZ, \rho_{\underline{k}})$ defined in Section 3.1, induces by restriction a G-equivariant endomorphism of $ind_{KZ}^{G}\rho_{\underline{k}}^0$, which we again denote by T.

The following Lemma is proved in [9, Lemma 3.3], but for sake of completeness we include here a proof of both isomorphisms.

Lemma 3.8. *There are isomorphisms of $\mathcal{O}_C$-algebras $\mathcal{H}_{\rho_{\underline{k}}^0}(KZ, G) \simeq \mathcal{O}_C[T]$ and $\mathcal{H}_{\rho_{\underline{k}}}(KZ, G) \simeq C[T]$.*

Proof. The space $V_{\rho_{\underline{k}}}$ is an absolutely irreducible $\mathfrak{g} \otimes_{\mathbb{Q}_p} C$-module, hence by Lemma 2.5, $\iota_{\rho_{\underline{k}}}$ is an isomorphism of C-algebras. Lemma 2.4 shows that there exists a unique morphism of C-algebras $u_{\rho_{\underline{k}}} : \mathcal{H}_C(KZ, G) \to \mathcal{H}_{\rho_{\underline{k}}}(KZ, G)$ making the following diagram commute

$$\begin{array}{ccc} \mathcal{H}_{KZ}(C) & \xrightarrow{\sim} & \mathcal{H}_C(KZ, G) \\ \downarrow{\scriptstyle \iota_{\rho_{\underline{k}}}} & & \downarrow{\scriptstyle u_{\rho_{\underline{k}}}} \\ \mathcal{H}_{KZ}(\rho_{\underline{k}}) & \xrightarrow{\sim} & \mathcal{H}_{\rho_{\underline{k}}}(KZ, G) \end{array} \tag{7}$$

By construction, this morphism is an isomorphism of C-algebras. Denote by $T_1 \in \mathcal{H}_C(KZ, G)$ the element corresponding to $\mathbf{1}_{KZ\alpha^{-1}KZ} \in \mathcal{H}_{KZ}(C)$ by Frobenius reciprocity.

If $\varphi \in \mathcal{H}_{KZ}(C)$, then as it has compact support, by the Cartan decomposition (1), it is supported on $\coprod_{i=0}^{n} KZ\alpha^{-i}KZ$ for some integer n. As φ is KZ-bi-invariant (recall that C is the trivial representation), its restriction to each $KZ\alpha^{-i}KZ$ is constant, hence we may write $\varphi = \sum_{i=0}^{n} \varphi_i \cdot \mathbf{1}_{KZ\alpha^{-i}KZ}$. Let $T_i \in \mathcal{H}_C(KZ, G)$ be the operator corresponding to $\mathbf{1}_{KZ\alpha^{-i}KZ}$ by Frobenius reciprocity. Then we see that the T_n's span $\mathcal{H}_C(KZ, G)$ over C. Geometrically, T_n is the operator associating to a vertex s the sum of the vertices at distance n from s: this is because

$$\mathbf{1}_{KZ\alpha^{-n}KZ} = \sum_{KZx \in KZ\backslash KZ\alpha^{-n}KZ} \mathbf{1}_{KZx} =$$

$$= \sum_{KZx \in KZ\backslash KZ\alpha^{-n}KZ} [x^{-1}, 1] = \sum_{KZx \in KZ\backslash KZ\alpha^{-n}KZ} x^{-1} \cdot [1, 1]$$

and then the $x^{-1}s_0$ are all distinct and give all vertices $s' \in \mathcal{T}_0$ such that s' is KZ-equivalent to $s_n = \alpha^{-n}s_0$. This means that (s_0, s') is equivalent to (s_0, s_n), which is precisely our assertion. From the geometrical description of T_n, one gets directly, since the tree $\mathcal{T}$ is $(q+1)$-regular, that

$$\begin{aligned} T_1^2 &= T_2 + (q+1)Id \\ T_1 T_{n-1} &= T_n + qT_{n-2} \qquad \forall n \geq 3 \end{aligned}$$

It follows that for all n, $T_n \in \mathcal{O}_C[T_1]$ is monic of degree n. In particular, $\mathcal{H}_C(KZ, G) \simeq C[T_1]$. Since $u_{\underline{\rho}_{\underline{k}}}(T_1) = T$, it follows that $\mathcal{H}_{\underline{\rho}_{\underline{k}}}(KZ, G) \simeq C[T]$.

Let us show that the restriction of this isomorphism to $\mathcal{H}_{\underline{\rho}^0_{\underline{k}}}(KZ, G)$ has image $\mathcal{O}_C[T]$.

As $T \in \mathcal{H}_{\underline{\rho}^0_{\underline{k}}}(KZ, G)$, clearly $\mathcal{O}_C[T]$ is contained in the image. Let $p(T) \in C[T]$ be a polynomial corresponding to an element in $\mathcal{H}_{\underline{\rho}^0_{\underline{k}}}(KZ, G)$.

Assume $\deg(p) = n$, and let a_n be the leading coefficient, i.e. $p(T) = a_n T^n + p_{n-1}(T)$, where $\deg(p_{n-1}) = n-1$. It follows that $p(T) = a_n T_n + q_{n-1}(T)$, for some q with $\deg(q_{n-1}) = n-1$.

We recall that T_n is the image under the natural isomorphisms of $\mathbf{1}_{KZ\alpha^{-n}KZ} \in H_{KZ}(C)$, which maps to $\mathbf{1}_{KZ\alpha^{-n}KZ} \cdot \underline{\rho}_{\underline{k}} \in H_{KZ}(\underline{\rho}_{\underline{k}})$, finally mapping to

$$T_n([g,v]) = \sum_{xKZ \in G/KZ} [gx, \mathbf{1}_{KZ\alpha^{-n}KZ}(x^{-1}) \underline{\rho}_{\underline{k}}(x^{-1})(v)] =$$

$$= \sum_{xKZ \in KZ\alpha^{-n}KZ/KZ} [gx, \underline{\rho}_{\underline{k}}(x^{-1})(v)]$$

Since $\alpha^n \in KZ\alpha^{-n}KZ$, and polynomials of order less than n are supported on $\coprod_{i=0}^{n-1} KZ\alpha^{-i}KZ$, it follows that for any $v \in \underline{\rho}_{\underline{k}}$, one has

$$(p(T)([1,v]))(\alpha^n) = (a_n T_n([1,v]))(\alpha^n) = a_n \underline{\rho}_{\underline{k}}(\alpha^{-n})(v) = a_n U^n_{\underline{k}}(v)$$

where the right most equality follows from (5).

In particular, taking $v = \bigotimes_{\sigma: F \hookrightarrow C} y_\sigma^{k_\sigma}$, we see that $v \in \underline{\rho}^0_{\underline{k}}$, hence $[1,v] \in ind_{KZ}^G \underline{\rho}^0_{\underline{k}}$. As we assume $p(T) \in \mathcal{H}_{\underline{\rho}^0_{\underline{k}}}(KZ, G) = End_{\mathcal{O}_C[G]}(ind_{KZ}^G \underline{\rho}^0_{\underline{k}})$, it follows that $p(T)([1,v]) \in ind_{KZ}^G \underline{\rho}^0_{\underline{k}}$, hence $a_n U^n_{\underline{k}}(v) = (p(T)([1,v]))(\alpha^n) \in \underline{\rho}^0_{\underline{k}}$. But, by definition of U, we see that $U_{\underline{k}}(v) = v$, hence $a_n v \in \underline{\rho}^0_{\underline{k}}$.

However, by definition of $\underline{\rho}^0_{\underline{k}}$, this is possible if and only if $a_n \in \mathcal{O}_C$. Therefore, we see that $a_n T^n \in \mathcal{O}_C[T]$, and it suffices to prove the claim for $p(T) - a_n T^n = p_{n-1}(T)$, which is a polynomial of degree less than n.

Proceeding by induction, where the induction basis consists of constant polynomials, which can be integral if and only if they belong to $\mathcal{O}_C$, we conclude that $p(T) \in \mathcal{O}_C[T]$. □

3.3. Formulas

We keep the notations of Sections 3.1 and 3.2. For $0 \leq m \leq n$, we denote by $[\cdot]_m : I_n \to I_m$ the "truncation" map, defined by:

$$\left[\sum_{i=0}^{n-1} \varpi^i [\mu_i]\right]_m = \begin{cases} \sum_{i=0}^{m-1} \quad \varpi^i [\mu_i] & m \geq 1 \\ 0 & m = 0 \end{cases}$$

For $\mu \in I_n$, we denote

$$\lambda_\mu = \frac{\mu - [\mu]_{n-1}}{\varpi^{n-1}} \in I_1$$

so that if $\mu = \sum_{i=0}^{n-1} \varpi^i [\mu_i]$, then $\lambda_\mu = [\mu_{n-1}]$.

We then have the following two results (see [3, 9]), where ψ denotes the function defined in Lemma 3.2.

Lemma 3.9. *Let $n \in \mathbb{N}, \mu \in I_n$, and let $v \in V_{\rho^0_{\underline{k}}}$. We have:*

$$T\left([g^0_{n,\mu}, v]\right) = T^+\left([g^0_{n,\mu}, v]\right) + T^-\left([g^0_{n,\mu}, v]\right)$$

where

$$T^+\left([g^0_{n,\mu}, v]\right) := \sum_{\lambda \in I_1} \left[g^0_{n+1,\mu+\varpi^n\lambda}, \left(\rho_{\underline{k}}(w) \circ \psi(\alpha^{-1}) \circ \rho_{\underline{k}}(w_\lambda)\right)(v)\right]$$

and

$$T^-\left([g^0_{n,\mu}, v]\right) := \begin{cases} \left[g^0_{n-1,[\mu]_{n-1}}, \left(\rho_{\underline{k}}(ww_{-\lambda_\mu}) \circ \psi(\alpha^{-1})\right)(v)\right] & n \geq 1 \\ [\alpha, \psi(\alpha^{-1})(v)] & n = 0 \end{cases}$$

Lemma 3.10. *Let $n \in \mathbb{N}, \mu \in I_n$, and let $v \in V_{\rho^0_{\underline{k}}}$. We have:*

$$T\left([g^1_{n,\mu}, v]\right) = T^+\left([g^1_{n,\mu}, v]\right) + T^-\left([g^1_{n,\mu}, v]\right)$$

where

$$T^+\left([g^1_{n,\mu}, v]\right) := \sum_{\lambda \in I_1} \left[g^1_{n+1,\mu+\varpi^n\lambda}, \left(\psi(\alpha^{-1}) \circ \rho_{\underline{k}}(w_\lambda w)\right)(v)\right]$$

and

$$T^-\left([g^1_{n,\mu}, v]\right) := \begin{cases} \left[g^1_{n-1,[\mu]_{n-1}}, \left(\rho_{\underline{k}}(w_{-\lambda_\mu}) \circ \psi(\alpha^{-1}) \circ \rho_{\underline{k}}(w)\right)(v)\right] & n \geq 1 \\ \left[Id, \left(\rho_{\underline{k}}(w) \circ \psi(\alpha^{-1}) \circ \rho_{\underline{k}}(w)\right)(v)\right] & n = 0 \end{cases}$$

By using the equality $g^1_{n,\mu} = \beta g^0_{n,\mu} w$, these two Lemmata yield the following two equalities:

$$\begin{aligned} T^+([g^1_{n,\mu}, v]) &= \beta T^+([g^0_{n,\mu}, \rho_{\underline{k}}(w)(v)]) \\ T^-([g^1_{n,\mu}, v]) &= \beta T^-([g^0_{n,\mu}, \rho_{\underline{k}}(w)(v)]) \end{aligned}$$

and also the following result

Corollary 3.11. *Let $n \in \mathbb{N}$, $\mu, \lambda \in I_n$, $i, j \in \{0,1\}$ and $v_1, v_2 \in V_{\rho^0_{\underline{k}}}$. If $i \neq j$ or if $\mu \neq \lambda$, then $T^+([g^i_{n,\mu}, v_1])$ and $T^+([g^j_{n,\lambda}, v_2])$ have disjoint supports.*

The following Lemma is a simple generalization of [3], Lemma 2.2.2.

Lemma 3.12. *Let $v = \sum_{\underline{0} \le \underline{i} \le \underline{k}} c_{\underline{i}} e_{\underline{k},\underline{i}} \in V_{\rho^0_{\underline{k}}}$ and $\lambda \in \mathcal{O}_F$. We have:*

$$\left(\rho_{\underline{k}}(w) \circ \psi(\alpha^{-1}) \circ \rho_{\underline{k}}(w_\lambda)\right)(v) = \sum_{\underline{0} \le \underline{j} \le \underline{k}} \left(\varpi^{\underline{j}} \sum_{\underline{j} \le \underline{i} \le \underline{k}} c_{\underline{i}} \binom{\underline{i}}{\underline{j}} (-\lambda)^{\underline{i}-\underline{j}} \right) e_{\underline{k},\underline{j}} \quad (8)$$

$$\left(\rho_{\underline{k}}(ww_\lambda) \circ \psi(\alpha^{-1})\right)(v) = \sum_{\underline{0} \le \underline{j} \le \underline{k}} \left(\sum_{\underline{j} \le \underline{i} \le \underline{k}} \varpi^{\underline{k}-\underline{i}} \binom{\underline{i}}{\underline{j}} c_{\underline{i}} (-\lambda)^{\underline{i}-\underline{j}} \right) e_{\underline{k},\underline{j}} \quad (9)$$

$$\psi(\alpha^{-1})(v) = \sum_{\underline{0} \le \underline{j} \le \underline{k}} \varpi^{\underline{k}-\underline{j}} c_{\underline{j}} e_{\underline{k},\underline{j}} \quad (10)$$

Proof. Equation (8) is proved in [9] and equation (10) is immediate. For equation (9), we note that by equation (4), we have for any $\sigma \in S$ and any $0 \le i_\sigma \le k_\sigma$:

$$\begin{aligned} (w \circ w_\lambda \circ U_{k_\sigma})(e_{k_\sigma,i_\sigma}) &= \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \left(\sigma(\varpi)^{k_\sigma - i_\sigma} e_{k_\sigma,i_\sigma}\right) = \\ &= \sigma(\varpi)^{k_\sigma - i_\sigma} \cdot x^{k_\sigma - i_\sigma} (y + \sigma(-\lambda)x)^{i_\sigma} = \\ &= \sigma(\varpi)^{k_\sigma - i_\sigma} \cdot \sum_{j_\sigma=0}^{i_\sigma} \binom{i_\sigma}{j_\sigma} \sigma(-\lambda)^{i_\sigma - j_\sigma} x^{k_\sigma - j_\sigma} y^{j_\sigma} = \\ &= \sigma(\varpi)^{k_\sigma - i_\sigma} \cdot \sum_{j_\sigma=0}^{i_\sigma} \binom{i_\sigma}{j_\sigma} \sigma(-\lambda)^{i_\sigma - j_\sigma} e_{k_\sigma,j_\sigma} \end{aligned}$$

Using equation (6), we deduce that

$$\begin{aligned} \left(\rho_{\underline{k}}(ww_\lambda) \circ \psi(\alpha^{-1})\right)(v) &= \sum_{\underline{0} \le \underline{i} \le \underline{k}} c_{\underline{i}} \cdot \bigotimes_{\sigma \in S} (w \circ w_\lambda \circ U_{k_\sigma})(e_{k_\sigma,i_\sigma}) = \\ &= \sum_{\underline{0} \le \underline{i} \le \underline{k}} c_{\underline{i}} \cdot \bigotimes_{\sigma \in S} \left(\sigma(\varpi)^{k_\sigma - i_\sigma} \cdot \sum_{j_\sigma=0}^{i_\sigma} \binom{i_\sigma}{j_\sigma} \sigma(-\lambda)^{i_\sigma - j_\sigma} e_{k_\sigma,j_\sigma} \right) = \end{aligned}$$

$$= \sum_{\underline{0}\leq\underline{i}\leq\underline{k}} c_{\underline{i}} \cdot \prod_{\sigma\in S} \sigma(\varpi)^{k_\sigma - i_\sigma} \cdot \sum_{\underline{0}\leq\underline{j}\leq\underline{i}} \prod_{\sigma\in S} \binom{i_\sigma}{j_\sigma} \cdot \prod_{\sigma\in S} \sigma(-\lambda)^{i_\sigma - j_\sigma} e_{\underline{k},\underline{j}} =$$

$$= \sum_{\underline{0}\leq\underline{j}\leq\underline{k}} \left(\sum_{\underline{j}\leq\underline{i}\leq\underline{k}} \varpi^{\underline{k}-\underline{i}} \cdot \binom{\underline{i}}{\underline{j}} \cdot c_{\underline{i}} \cdot (-\lambda)^{\underline{i}-\underline{j}} \right) e_{\underline{k},\underline{j}}$$

□

This leads to the following corollary, which is a simple generalization of [3], Corollary 2.2.3.

Corollary 3.13. *Let $m \in \mathbb{Z}_{>0}$, $a \in C$, and for any $\mu \in I_m$ (resp. $\mu \in I_{m-1}$, resp. $\mu \in I_{m+1}$), $v_\mu^m = \sum_{\underline{0}\leq\underline{i}\leq\underline{k}} c_{\underline{i},\mu}^m \cdot e_{\underline{k},\underline{i}}$ (resp. $v_\mu^{m-1} = \sum_{\underline{0}\leq\underline{i}\leq\underline{k}} c_{\underline{i},\mu}^{m-1} \cdot e_{\underline{k},\underline{i}}$, resp. $v_\mu^{m+1} = \sum_{\underline{0}\leq\underline{i}\leq\underline{k}} c_{\underline{i},\mu}^{m+1} \cdot e_{\underline{k},\underline{i}}$) an element of $\rho_{\underline{k}}$. We denote*

$$f_m = \sum_{\mu\in I_m} [g_{m,\mu}^0, v_\mu^m]$$

$$f_{m-1} = \sum_{\mu\in I_{m-1}} [g_{m-1,\mu}^0, v_\mu^{m-1}]$$

$$f_{m+1} = \sum_{\mu\in I_{m+1}} [g_{m+1,\mu}^0, v_\mu^{m+1}]$$

Then

$$T^-(f_{m+1}) + T^+(f_{m-1}) - af_m = \sum_{\mu\in I_m} \left[g_{m,\mu}^0, \sum_{\underline{0}\leq\underline{j}\leq\underline{k}} C_{\underline{j},\mu}^m \cdot e_{\underline{k},\underline{j}} \right]$$

where

$$C_{\underline{j},\mu}^m = \sum_{\underline{j}\leq\underline{i}\leq\underline{k}} \varpi^{\underline{k}-\underline{i}} \binom{\underline{i}}{\underline{j}} \cdot \sum_{\lambda\in k_F} c_{\underline{i},\mu+\varpi^m[\lambda]}^{m+1} \cdot [\lambda]^{\underline{i}-\underline{j}} + \varpi^{\underline{j}} \cdot \sum_{\underline{j}\leq\underline{i}\leq\underline{k}} c_{\underline{i},[\mu]_{m-1}}^{m-1} \binom{\underline{i}}{\underline{j}} (-\lambda_\mu)^{\underline{i}-\underline{j}} - ac_{\underline{j},\mu}^m \tag{11}$$

4. A Criterion for Separability

4.1. The main result

We adhere to the notations of Sections 3.2 and 3.3 and fix an embedding $\iota : F \hookrightarrow C$. Denote

$$S^+ = \{\sigma \in S \mid k_\sigma \neq 0\} \subseteq S$$

We partition S^+ with respect to the action of $\sigma \in S^+$ on the residue field of F. More precisely, for any $l \in \{0, \ldots, f-1\}$, we let

$$J_l = \left\{\sigma \in S^+ \mid \sigma(\lambda) = \iota \circ \varphi_0^l(\lambda) \quad \forall \lambda \in I_1\right\}$$

where $I_1 = \{[\zeta] \mid \zeta \in \kappa_F\}$. In particular, we remark that

$$\coprod_{l=0}^{f-1} J_l = S^+, \quad \forall l \in \{0, \dots, f-1\} \quad |J_l| \leq e.$$

For any integer $i \in \mathbb{Z}$, we denote by $\bar{i}$ the unique representative of $i \mod f$ in $\{0, \dots, f-1\}$. We also let, for any $\sigma \in J_l$, $\gamma_\sigma := l$ and

$$v_\sigma = \inf \left\{ i \mid 1 \leq i \leq f, J_{\overline{l+i}} \neq \emptyset \right\}$$

that is the minimal power of Frobenius φ_0 needed to pass from J_l to another nonempty J_k.

Let $a \in \mathfrak{p}_C$. We let

$$\Pi_{\underline{k},a} = \frac{ind_{KZ}^G \underline{\rho}_{\underline{k}}}{(T-a)(ind_{KZ}^G \underline{\rho}_{\underline{k}})}.$$

This is a locally algebraic representation of G, which can be realized as the tensor product of an algebraic representation with a smooth representation. More precisely, we have the following result, which is stated in [9].

Proposition 4.1. *Let $u_\sigma = \frac{k_\sigma}{2}$ for any $\sigma \in S$.*
(i) If $a \notin \{\pm((q+1)\varpi^{\underline{u}}\}$, then $\Pi_{\underline{k},a}$ is algebraicly irreducible and

$$\Pi_{\underline{k},a} \simeq \underline{\rho}_{\underline{k}} \otimes Ind_B^G(nr(\lambda_1^{-1}) \otimes nr(\lambda_2^{-1}))$$

where λ_1, λ_2 satisfy

$$\lambda_1 \lambda_2 = \varpi^{\underline{k}}, \quad \lambda_1 + q\lambda_2 = a$$

(ii) If $a \in \{\pm((q+1)\varpi^{\underline{u}}\}$, then we have a short exact sequence

$$0 \to \underline{\rho}_{\underline{k}} \otimes St_G \otimes (nr(\delta) \circ \det) \to \Pi_{\underline{k},a} \to \underline{\rho}_{\underline{k}} \otimes (nr(\delta) \circ \det) \to 0$$

where $St_G = C^0(\mathbf{P}^1(F), C)/\{constants\}$ denotes the Steinberg representation of G and where $\delta = (q+1)/a$.

As in [9], we define

$$\Theta_{\underline{k},a} = \operatorname{Im}\left(ind_{KZ}^G \underline{\rho}_{\underline{k}}^0 \to \Pi_{\underline{k},a} \right)$$

which is the same as

$$\Theta_{\underline{k},a} = \frac{ind_{KZ}^G \underline{\rho}_{\underline{k}}^0}{ind_{KZ}^G \underline{\rho}_{\underline{k}}^0 \cap (T-a)(ind_{KZ}^G \underline{\rho}_{\underline{k}})}.$$

This is a lattice in $\Pi_{\underline{k},a}$ and, since $ind_{KZ}^G \rho_{\underline{k}}^0$ is a finitely generated $\mathcal{O}_C[G]$-module, we see that $\Theta_{\underline{k},a}$ is also a finitely generated $\mathcal{O}_C[G]$-module.

Now, the Breuil Schneider conjecture 1.2 asserts that $\underline{\rho}_{\underline{k}} \otimes \pi$ admits a G-invariant norm.

By [11, Prop. 1.17], this is equivalent to the existence of a separated lattice, and even to any finitely generated lattice being separated.

The following conjecture is then a restatement of the Breuil-Schneider conjecture.

Conjecture 4.2. *The $\mathcal{O}_C$-module $\Theta_{\underline{k},a}$ does not contain any C-line (it is separated).*

We also recall that Breuil, in [3] proves the conjecture for $F = \mathbb{Q}_p$ and $k < 2p-1$, and that de Ieso, in [9], proves it when $|J_l| \leq 1$ for all $l \in \{0, \ldots, f-1\}$ and for any $\sigma \in S^+$, $k_\sigma + 1 \leq p^{v_\sigma}$.

The idea, as in [3], is to reduce the problem to a statement which we can prove inductively, sphere by sphere.

As we shall use that idea repeatedly, we introduce a related definition. Abusing notation, we denote by $B_N \subseteq ind_{KZ}^G \underline{\rho}_{\underline{k}}$ the set of functions supported in $B_N = B_N^0 \coprod B_{N-1}^1$, where $B_N^0 = \coprod_{M \leq N} S_M^0$, $B_N^1 = \coprod_{M \leq N} S_M^1$, and we have defined

$$S_M^0 = I\alpha^{-M}KZ, \quad S_M^1 = I\beta\alpha^{-M}KZ$$

We also recall that B^0, B^1 denote the sets of functions supported on $\bigcup_N B_N^0, \bigcup_N B_N^1$, respectively.

Definition 4.3. Let $\underline{k} \in \mathbb{N}^S$, and let $a \in \mathcal{O}_C$. We say that the pair $(\underline{k}, a)$ is *separated* if for all $N \in \mathbb{Z}_{>0}$ large enough, there exists a constant $\epsilon \in \mathbb{Z}_{\geq 0}$ depending only on $N, \underline{k}, a$ such that for all $n \in \mathbb{Z}_{\geq 0}$, and all $f \in B^0$

$$(T-a)(f) \in B_N + \varpi^n ind_{KZ}^G \underline{\rho}_{\underline{k}}^0 \Rightarrow f \in B_{N-1} + \varpi^{n-\epsilon} ind_{KZ}^G \underline{\rho}_{\underline{k}}^0 \qquad (12)$$

Remark 4.4. We slightly abuse notation here, as $\varpi \notin C$, but as $v_F(\sigma(\varpi)) = v_F(\varpi) = 1$ for all $\sigma \in S$, one may choose any embedding $\sigma : F \hookrightarrow C$, and consider $\sigma(\varpi)^n$ instead.

The upshot is that we have the following result.

Theorem 4.5. *Let $\underline{k} \in \mathbb{N}^S$, let $a \in \mathcal{O}_C$. If $(\underline{k}, a)$ is separated, then $\Theta_{\underline{k},a}$ is separated.*

Proof. First, note that if (12) holds for all $f \in ind_{KZ}^G \underline{\rho}_{\underline{k}}$, then the proof of [3], Corollary 4.1.2 shows that $\Theta_{\underline{k},a}$ is separated.

Next, for an arbitary $f \in ind_{KZ}^G \underline{\rho}_{\underline{k}}$, write $f = f^0 + f^1$ with $f^0 \in B^0$ and $f^1 \in B^1$. Then by the formulas in Lemma 3.9 and Lemma 3.10, it follows that

$$\text{supp}\left((T-a)(f^0)\right) \cap \text{supp}\left((T-a)(f^1)\right) \subseteq S_0 = B_0 \subseteq B_N$$

If we assume that

$$(T-a)(f^0)+(T-a)(f^1)=(T-a)(f)\in B_N+\varpi^n ind_{KZ}^G \rho_{\underline{k}}^0$$

it follows that both $(T-a)(f^0)\in B_N+\varpi^n ind_{KZ}^G \rho_{\underline{k}}^0$ and $(T-a)(f^1)\in B_N+\varpi^n ind_{KZ}^G \rho_{\underline{k}}^0$.

Since $f^0\in B^0$ and $(\underline{k},a)$ is separated, it follows that $f^0\in B_{N-1}+\varpi^{n-\epsilon} ind_{KZ}^G \rho_{\underline{k}}^0$.

Moreover, since T is G-equivariant, and $\varpi^{\mathbb{Z}}\cdot Id$ acts trivially, we see that

$$\beta(T-a)(\beta f^1)=(T-a)(f^1)\in B_N+\varpi^n ind_{KZ}^G \rho_{\underline{k}}^0$$

Since β acts by translation, it does not affect the values of the function, and since $\beta B_N=B_N$, it follows that

$$(T-a)(\beta f^1)\in B_N+\varpi^n ind_{KZ}^G \rho_{\underline{k}}^0$$

with $\beta f^1\in B^0$. Since $(\underline{k},a)$ is separated, we get $\beta f^1\in B_{N-1}+\varpi^{n-\epsilon} ind_{KZ}^G \rho_{\underline{k}}^0$, hence $f^1\in B_{N-1}+\varpi^{n-\epsilon} ind_{KZ}^G \rho_{\underline{k}}^0$.

In conclusion

$$f=f^0+f^1\in B_{N-1}+\varpi^{n-\epsilon} ind_{KZ}^G \rho_{\underline{k}}^0$$

as claimed. □

It therefore remains to show that certain pairs $(\underline{k},a)$ are separated.

In this section, we will prove the following theorem:

Theorem 4.6. *Assume that $|S^+|=1$, denote by σ the unique element in S^+, and and let $k=k_\sigma=d\cdot q+r$, with $0\le r<q$. Assume that one of the following three conditions is satisfied:*

(i) $k\le\frac{1}{2}q^2$ with $r<q-d$ and $v_F(a)\in[0,1]$.

(ii) $k\le\frac{1}{2}q^2$ with $2v_F(a)-1\le r<q-d$ and $v_F(a)\in[1,e]$.

(iii) $k\le\min\left(p\cdot q-1,\frac{1}{2}q^2\right)$, $d-1\le r$ and $v_F(a)\ge d$.

Then $(\underline{k},a)$ is separated.

Corollary 4.7. *Under the above conditions, $\Theta_{\underline{k},a}$ is separated, hence $\Pi_{\underline{k},a}$ admits an invariant norm.*

Since our assumptions include the fact that $|S^+|=1$, we may proceed with the following notational simplifications.

We assume that C contains F, and let $\sigma=\iota:F\hookrightarrow C$ be the natural inclusion.

We may further let $k=k_\sigma$ stand for the multi-index $\underline{k}$ corresponding to k, and similarly for all multi-indices.

Before we prove the theorems, let us first prove the following useful lemmata, which we will employ later on.

Lemma 4.8. *Let κ be a finite field of characteristic p containing $\mathbb{F}_q$. Consider a polynomial $h \in \kappa[x]$, such that*

$$h(x+\lambda) \in x^j \cdot \kappa[x] \quad \forall \lambda \in \mathbb{F}_q$$

Then

$$h(x) \in (x^q - x)^j \cdot \kappa[x]$$

Proof. We will prove the Lemma by induction on j. For $j = 1$, $h(x+\lambda) \in x \cdot \kappa[x]$ implies that $h(\lambda) = 0$ for all $\lambda \in \mathbb{F}_q$, hence $x^q - x \mid h(x)$, as claimed.

In general, $h(x+\lambda) \in x^j \cdot \kappa[x] \subseteq x \cdot \kappa[x]$ for all $\lambda \in \mathbb{F}_q$, hence $h(x) = (x^q - x) \cdot g(x)$ for some $g(x) \in \kappa[x]$, by the $j = 1$ case. But $\gcd(x^q - x, x^j) = x$, hence we get

$$h(x+\lambda) = (x^q - x) \cdot g(x+\lambda) \in x^j \cdot \kappa[x] \Rightarrow g(x+\lambda) \in x^{j-1} \cdot \kappa[x]$$

for all $\lambda \in \mathbb{F}_q$.

By the induction hypothesis, it follows that $g(x) \in (x^q - x)^{j-1} \cdot \kappa[x]$, hence $h(x) \in (x^q - x)^j \cdot \kappa[x]$. $\square$

Lemma 4.9. *Let $k, d \in \mathbb{N}$. Let $h(x) = \sum_{i=0}^k c_i x^i \in \mathcal{O}_C[x]$ be such that for all $0 \le j \le d$, and all $\lambda \in \mathbb{F}_q$, we have*

$$\sum_{i=j}^k \binom{i}{j} c_i [\lambda]^{i-j} \in \varpi_C \mathcal{O}_C$$

where $[\lambda] \in \mathcal{O}_F \hookrightarrow \mathcal{O}_C$ is the Teichmüller representative of λ. Then $h(x) \in (x^q - x)^{d+1} \cdot \mathcal{O}_C[x] + \varpi_C \cdot \mathcal{O}_C[x]$.

Proof. By our assumption, since

$$h(x+[\lambda]) = \sum_{i=0}^k c_i (x+[\lambda])^i = \sum_{i=0}^k c_i \sum_{j=0}^i \binom{i}{j} x^j [\lambda]^{i-j} = \\ = \sum_{j=0}^k \left(\sum_{i=j}^k \binom{i}{j} c_i [\lambda]^{i-j} \right) x^j \tag{13}$$

we see that

$$h(x+[\lambda]) \in \left(x^{d+1}, \varpi_C\right) \quad \forall \lambda \in \mathbb{F}_q$$

Equivalently, considering the image in $k_C = \mathcal{O}_C / \varpi_C \mathcal{O}_C$, we have $\overline{h} \in \kappa_C[x]$ of degree at most k, satisfying

$\overline{h}(x+\lambda) \in (x^{d+1})$ for all $\lambda \in \mathbb{F}_q$.

By Lemma 4.8, we see that $\overline{h}(x) \in (x^q - x)^{d+1} \cdot \kappa_C[x]$, hence $h(x) \in \left((x^q - x)^{d+1}, \varpi_C\right)$. This establishes the Lemma. $\square$

Lemma 4.10. *Let $n, k, d \in \mathbb{N}$. Let $f(x) = \sum_{i=0}^{k} c_i x^i \in \mathcal{O}_C[x]$ be such that for all $0 \leq j \leq d$ and all $\lambda \in \mathbb{F}_q$ we have*

$$\sum_{i=j}^{k} \binom{i}{j} c_i [\lambda]^{i-j} \in \varpi_C^n \mathcal{O}_C$$

where $[\lambda] \in \mathcal{O}_F \hookrightarrow \mathcal{O}_C$ is the Teichmüller representative of λ. Then $f(x) \in (x^q - x)^{d+1} \cdot \mathcal{O}_C[x] + \varpi_C^n \cdot \mathcal{O}_C[x]$.

Proof. By induction on n. For $n = 1$, this is Lemma 4.9. Assume it holds for $n-1$, and let us prove it for n.
Since $\varpi_C^n \mathcal{O}_C \subseteq \varpi_C^{n-1} \mathcal{O}_C$, the induction hypothesis implies that $f(x) \in \left((x^q - x)^{d+1}, \varpi_C^{n-1}\right)$, so we may write

$$f(x) = (x^q - x)^{d+1} \cdot g(x) + \varpi_C^{n-1} \cdot h(x)$$

By (13), our assumption implies that

$$f(x + [\lambda]) \in \left(x^{d+1}, \varpi_C^n\right) \quad \forall \lambda \in \mathbb{F}_q$$

substituting in the above equation, we get

$$((x + [\lambda])^q - (x + [\lambda]))^{d+1} \cdot g(x + [\lambda]) + \varpi_C^{n-1} \cdot h(x + [\lambda]) \in (x^{d+1}, \varpi_C^n)$$

But

$$(x+[\lambda])^q - (x+[\lambda]) = \sum_{i=0}^{q} \binom{q}{i} [\lambda]^{q-i} x^i - x - [\lambda] = \sum_{i=1}^{q} \binom{q}{i} [\lambda]^{q-i} x^i - x \in x \cdot \mathcal{O}_C[x]$$

since $[\lambda]^q = [\lambda]$ for all $\lambda \in \mathbb{F}_q$. This shows that $((x + [\lambda])^q - (x + [\lambda]))^{d+1} \in (x^{d+1}) \subseteq (x^{d+1}, \varpi_C^n)$, hence

$$\varpi_C^{n-1} \cdot h(x + [\lambda]) \in (x^{d+1}, \varpi_C^n) \quad \forall \lambda \in \mathbb{F}_q$$

which implies that

$$h(x + [\lambda]) \in (x^{d+1}, \varpi_C) \quad \forall \lambda \in \mathbb{F}_q$$

Considering the reduction modulo ϖ_C, by Lemma 4.8, it follows that $h(x) \in \left((x^q - x)^{d+1}, \varpi_C\right)$, hence

$$f(x) \in \left((x^q - x)^{d+1}\right) + \varpi_C^{n-1} \cdot \left((x^q - x)^{d+1}, \varpi_C\right) = \left((x^q - x)^{d+1}, \varpi_C^n\right)$$

establishing the claim. □

Lemma 4.11. *Let $n \in \mathbb{Z}$, $k, d \in \mathbb{N}$. Let $f(x) = \sum_{i=0}^{k} c_i x^i \in C[x]$ be such that for all $0 \le j \le d$ and all $\lambda \in \mathbb{F}_q$ we have*

$$\sum_{i=j}^{k} \binom{i}{j} c_i [\lambda]^{i-j} \in \varpi_C^n \mathcal{O}_C$$

where $[\lambda] \in \mathcal{O}_F \hookrightarrow \mathcal{O}_C$ is the Teichmüller representative of λ. Then $f(x) \in (x^q - x)^{d+1} \cdot C[x] + \varpi_C^n \cdot \mathcal{O}_C[x]$.

Proof. Let $L = \min_{0 \le i \le k} v_C(c_i)$. Consider $g(x) = \varpi_C^{-L} \cdot f(x) \in \mathcal{O}_C[x]$. If $n \le L$, then as $f(x) \in \varpi_C^L \mathcal{O}_C[x] \subseteq \varpi_C^n \mathcal{O}_C[x]$, we are done.
Else, $g(x)$ satisfies for all $0 \le j \le d$ and all $\lambda \in \mathbb{F}_q$

$$\sum_{i=j}^{k} \binom{i}{j} \varpi_C^{-L} \cdot c_i [\lambda]^{i-j} \in \varpi_C^{n-L} \mathcal{O}_C$$

with $n - L \ge 1$, hence by Lemma 4.10, $g(x) \in (x^q - x)^{d+1} \cdot \mathcal{O}_C[x] + \varpi_C^{n-L} \cdot \mathcal{O}_C[x]$, hence $f(x) \in (x^q - x)^{d+1} \cdot C[x] + \varpi_C^n \cdot \mathcal{O}_C[x]$. □

Lemma 4.12. *Let $n \in \mathbb{Z}$ and let $k \in \mathbb{N}$. Let $d = \lfloor k/q \rfloor$. Let $(c_i)_{i=0}^k$ be a sequence in C such that for all $0 \le j \le d$, and all $\lambda \in \mathbb{F}_q$, we have*

$$\sum_{i=j}^{k} \binom{i}{j} c_i [\lambda]^{i-j} \in \pi_C^n \mathcal{O}_C$$

where $[\lambda] \in \mathcal{O}_F \hookrightarrow \mathcal{O}_C$ is the Teichmüller representative of λ. Then $c_i \in \varpi_C^n \mathcal{O}_C$ for all $0 \le i \le k$.

Proof. By Lemma 4.11, we see that $f(x) = \sum_{i=0}^{k} c_i x^i \in (x^q - x)^{d+1} \cdot C[x] + \varpi_C^n \mathcal{O}_C[x]$, but $\deg(f) \le k < q(d+1)$, hence $f(x) \in \varpi_C^n \mathcal{O}_C[x]$. This establishes the Lemma. □

Lemma 4.13. *Let $k, d \in \mathbb{N}$. Let $f(x) = \sum_{i=0}^{k} c_i x^i \in C[x]$ and let $n \in \mathbb{Z}$. Assume that for all $0 \le j \le d$, and all $\lambda \in \mathbb{F}_q$, we have*

$$\sum_{i=j}^{k} \binom{i}{j} c_i [\lambda]^{i-j} \in \varpi_C^n \mathcal{O}_C$$

where $[\lambda] \in \mathcal{O}_F \hookrightarrow \mathcal{O}_C$ is the Teichmüller representative of λ. Then

$$c_i \in \varpi_C^n \mathcal{O}_C \quad \forall 0 \le i \le d$$

$$\sum_{l=d}^{\lfloor \frac{k-j}{q-1} \rfloor} \binom{l}{d} \cdot c_{j+l(q-1)} \in \varpi_C^n \mathcal{O}_C \quad \forall d+1 \le j \le d+q-1 \tag{14}$$

Proof. By Lemma 4.11, we see that $f(x) \in (x^q - x)^{d+1} \cdot C[x] + \varpi_C^n \mathcal{O}_C[x]$. We proceed by reducing $f(x)$ modulo $(x^q - x)^{d+1}$.

In order to do so, we first have to understand the reduction of a general monomial x^t modulo $(x^q - x)^{d+1}$.

We prove, by induction on s, that for every $0 \le s \le \left\lfloor \frac{t-d-1}{q-1} \right\rfloor - d - 1$ and every $t \ge q(d+1)$ we have

$$x^t \equiv \sum_{l=1}^{d+1} (-1)^{l+1} \binom{d+1+s}{l+s} \cdot \binom{l+s-1}{s} x^{t-(l+s)(q-1)} \mod (x^q - x)^{d+1} \quad (15)$$

Indeed, for $s = 0$, this is simply a restatement of the binomial expansion, as

$$\begin{aligned} x^t &= x^{t-(d+1)\cdot q} \cdot x^{(d+1)\cdot q} \equiv x^{t-(d+1)\cdot q} \cdot \left(x^{(d+1)q} - (x^q - x)^{d+1} \right) = \\ &= x^{t-(d+1)\cdot q} \cdot \left(x^{(d+1)q} - \sum_{l=0}^{d+1} (-1)^l \binom{d+1}{l} \cdot (x^q)^{(d+1)-l} \cdot x^l \right) = \\ &= x^{t-(d+1)\cdot q} \cdot \sum_{l=1}^{d+1} (-1)^{l+1} \binom{d+1}{l} x^{(d+1)\cdot q - l(q-1)} = \\ &= \sum_{l=1}^{d+1} (-1)^{l+1} \binom{d+1}{l} x^{t-l(q-1)} \mod (x^q - x)^{d+1} \end{aligned}$$

Assume it holds for $s - 1$, and let us prove it holds for s.
By the induction hypothesis

$$x^t \equiv \sum_{l=1}^{d+1} (-1)^{l+1} \binom{d+s}{l+s-1} \binom{l+s-2}{s-1} x^{t-(l+s-1)(q-1)} \mod (x^q - x)^{d+1} \quad (16)$$

Since $s \le \left\lfloor \frac{t-d-1}{q-1} \right\rfloor - d - 1$, we see that

$$(q-1)(d+1+s) \le t - (d+1) \Rightarrow t - s(q-1) \ge q(d+1)$$

This implies, by the case $s = 0$, that

$$x^{t-s(q-1)} \equiv \sum_{l=1}^{d+1} (-1)^{l+1} \binom{d+1}{l} \cdot x^{t-(l+s)(q-1)} \mod (x^q - x)^{d+1}$$

Substituting in (16) we get

$$\begin{aligned} x^t \quad &\equiv \quad \binom{d+s}{s} \cdot \sum_{l=1}^{d+1} (-1)^{l+1} \binom{d+1}{l} \cdot x^{t-(l+s)(q-1)} + \\ &+ \sum_{l=2}^{d+1} (-1)^{l+1} \binom{d+s}{l+s-1} \binom{l+s-2}{s-1} x^{t-(l+s-1)(q-1)} = \end{aligned}$$

$$= \sum_{l=1}^{d+1} (-1)^{l+1} \left(\binom{d+s}{s}\binom{d+1}{l} - \binom{d+s}{l+s}\binom{l+s-1}{s-1} \right) x^{t-(l+s)(q-1)}$$

Calculation yields

$$\binom{d+s}{s}\binom{d+1}{l} - \binom{d+s}{l+s}\binom{l+s-1}{s-1} =$$

$$= \frac{(d+s)!(d+1)!}{s!d!l!(d+1-l)!} - \frac{(d+s)!(l+s-1)!}{(l+s)!(d-l)!l!(s-1)!} =$$

$$= \frac{(d+s)!\cdot(d+1)\cdot(l+s)}{(l+s)\cdot s!l!(d+1-l)!} - \frac{(d+s)!\cdot s\cdot(d+1-l)}{(l+s)\cdot(d+1-l)!l!s!} =$$

$$= \frac{(d+s)!}{(l+s)\cdot s!l!(d+1-l)!}\cdot((d+1)l+(d+1)s-(d+1)s+sl) =$$

$$= \frac{(d+s+1)!}{(l+s)\cdot s!(l-1)!(d+1-l)!} =$$

$$= \frac{(d+1+s)!}{(l+s)!(d+1-l)!}\cdot\frac{(l+s-1)!}{s!(l-1)!} = \binom{d+1+s}{l+s}\binom{l+s-1}{s}$$

establishing the identity (15).

It now follows from (15), by letting $t = j + l(q-1)$ and $s = l - d - 1$, that

$$f(x) = \sum_{i=0}^{k} c_i x^i = \sum_{i=0}^{d} c_i x^i + \sum_{j=d+1}^{d+q-1} \sum_{l=0}^{\lfloor \frac{k-j}{q-1} \rfloor} c_{j+l(q-1)} x^{j+l(q-1)} \equiv$$

$$\equiv \sum_{i=0}^{d} c_i x^i + \sum_{j=d+1}^{d+q-1} \left(\sum_{l=0}^{d} c_{j+l(q-1)} x^{j+l(q-1)} + \sum_{l=d+1}^{\lfloor \frac{k-j}{q-1} \rfloor} c_{j+l(q-1)} \sum_{r=1}^{d+1} \gamma_{r,l,d} \cdot x^{j-(r-d-1)(q-1)} \right) =$$

$$= \sum_{i=0}^{d} c_i x^i + \sum_{j=d+1}^{d+q-1} \sum_{l=0}^{d} \left(c_{j+l(q-1)} + \sum_{m=d+1}^{\lfloor \frac{k-j}{q-1} \rfloor} \delta_{m,l,d} \cdot c_{j+m(q-1)} \right) x^{j+l(q-1)} \mod (x^q - x)^{d+1}$$

where

$$\gamma_{r,l,d} = (-1)^{r+1} \binom{l}{r+l-d-1} \cdot \binom{r+l-d-2}{l-d-1}$$

and

$$\delta_{m,l,d} = (-1)^{d-l} \binom{m}{l} \binom{m-l-1}{d-l}.$$

As this is a polynomial of degree less than $q(d+1)$, and we know that $f(x) \in (x^q - x)^{d+1} \cdot C[x] + \varpi_C^n \cdot \mathcal{O}_C[x]$, it follows that it must lie in $\varpi_C^n \cdot \mathcal{O}_C[x]$.

In particular, $c_i \in \varpi_C^n \mathcal{O}_C$ for all $0 \le i \le d$, and looking at the coefficient of $x^{j+d(q-1)}$ yields (14), as claimed. □

Lemma 4.14. *Let q be a power of a prime number p. Let $k = d \cdot q + r$ be such that $d < q$ and $0 \leq r < q - d$. Then for any $0 \leq i \leq r$, any $0 \leq j \leq d$ and any $j + 1 \leq l \leq d$, one has $p \mid \binom{k-i}{k-j-l(q-1)}$.*

Proof. Since $i \leq r < q$, we know that $0 \leq r - i < q$ and $d < q$, so that $k - i = d \cdot q + (r - i)$ is the base q representation of $k - i$.
Since $j + 1 \leq l \leq d$, one has $1 \leq r + 1 \leq r + l - j \leq r + l \leq r + d < q$ and it follows that

$$k - j - l(q - 1) = d \cdot q + r - l \cdot q + l - j = (d - l) \cdot q + (r + l - j)$$

is the base q representation of $k - j - l(q - 1)$.
Finally, by Kummer's Theorem on binomial coefficients, as for any $l \geq j + 1$ and any $i, j \geq 0$,

$$r + l - j \geq r + 1 > r \geq r - i$$

there is at least one digit in the base p representation of $r + l - j$, which is larger than the corresponding one in the base p representation of $r - i$, hence

$$p \mid \binom{k - i}{k - j - l(q - 1)}$$

establishing the result. □

Lemma 4.15. *Let $a \in \mathbb{Z}$. The matrix $A = A_m(a) \in \mathbb{Z}^{m \times m}$ with entries $(A_{li})_{l,i=1}^m = \binom{a+l}{i-1}$ satisfies* $\det A = 1$.

Proof. We prove it by induction on m. For $m = 1$, this is the matrix (1), which is nonsingular.
Note that for any $2 \leq l \leq m$, and any $1 \leq i \leq m$, one has

$$\binom{a + l}{i - 1} - \binom{a + l - 1}{i - 1} = \binom{a + l - 1}{i - 2}$$

where $\binom{k}{-1} = 0$.
Therefore, subtracting from each row its preceding row, we obtain the matrix B, with $B_{1i} = A_{1i}$ for all $1 \leq i \leq m$, and $B_{li} = \binom{a+l-1}{i-2}$.
By the induction hypothesis, the matrix $(B_{li})_{l,i=2}^m$ is in fact $A_{m-1}(a)$, $\det(B_{li})_{l,i=2}^m = 1$. But, as $B_{l1} = 0$ for all $l \geq 2$ and $B_{11} = 1$, it follows that $\det A = \det B = 1$. □

Corollary 4.16. *Let $a \in \mathbb{Z}$, $m \in \mathbb{N}$. Let $t \in \{2, \ldots, m\}$. Consider the matrix $A \in \mathbb{Z}^{m \times m}$ with entries*

$$A_{li} = \begin{cases} \binom{a+l}{i-1} & t \leq i \leq m \\ \binom{a+l+1}{i-1} & 1 \leq i < t \end{cases} \quad \forall l \in \{1, 2, \ldots, m\}$$

Then $\det A = 1$.

Proof. This matrix is obtained from the one in Lemma 4.15 by adding each of the first $t-2$ columns to its subsequent column, since

$$\binom{a+l+1}{i-1} = \binom{a+l}{i-1} + \binom{a+l}{i-2}$$

As these operations do not affect the determinant, the result follows. □

Corollary 4.17. *Let $k \in \mathbb{N}$. Write $k = d \cdot q + r$, with $1 \leq d < p$, $0 \leq r < q$ and assume that $d - 1 \leq r$. Let $1 \leq m \leq d$. Then the matrix $A \in \mathbb{F}_p^{m \times m}$ with entries $(A_{il})_{i,l=1}^m = \binom{k-i+1}{m+l(q-1)}$, is nonsingular.*

Proof. For any $1 \leq l \leq m$, we note that $m + l(q-1) = lq + (m-l)$, hence (as $d < q$ and $r - i + 1 \geq r - d + 1 \geq 0$) by Lucas' Theorem

$$\binom{k-i+1}{m+l(q-1)} = \binom{dq+r-i+1}{lq+(m-l)} \equiv \binom{d}{l} \cdot \binom{r-i+1}{m-l} \mod p$$

Since $1 \leq l \leq d < p$, we get that the $\binom{d}{l}$ are nonzero mod p, hence we can divide the l-th column by the appropriate multiplier without affecting the singularity of A, call the resulting matrix B.

Then $B_{il} = \binom{r-i+1}{m-l}$, which up to rearranging rows and cloumns, is the matrix from Lemma 4.15, hence nonsingular. ■ □

4.3. The case $v_F(a) \geq \left\lfloor \frac{k}{q} \right\rfloor$

In this section, we will prove the following theorem, which will establish (*iii*) in Theorem 4.6.

Theorem 4.18. *Let $0 \leq k \leq \min\left(p \cdot q - 1, \frac{q^2}{2}\right)$. Assume further that $k = dq + r$ with $d - 1 \leq r < q$. Let $a \in \mathcal{O}_C$ be such that $v_F(a) \geq d$, and let $N \in \mathbb{Z}_{>0}$. There exists a constant $\epsilon \in \mathbb{Z}_{\geq 0}$ depending only on N, k, a such that for all $n \in \mathbb{Z}_{\geq 0}$, and all $f \in ind_{KZ}^G \underline{\rho}_k$*

$$(T-a)(f) \in B_N + \varpi^n ind_{KZ}^G \underline{\rho}_k^0 \Rightarrow f \in B_{N-1} + \varpi^{n-\epsilon} ind_{KZ}^G \underline{\rho}_k^0$$

Proof. As before, we may assume that $f = \sum_{m=0}^M f_m$ where $f_m \in S_{N+m}^0$, and denote $f_m = 0$ for $m > M$. Looking at S_{N+m}, we have the equations

$$T^-(f_{m+1}) + T^+(f_{m-1}) - af_m \in \varpi^n ind_{KZ}^G \underline{\rho}_k^0 \quad 1 \leq m \leq M+1$$

We shall prove the theorem with $\epsilon = d$.

Assume, by descending induction on m, that $f_m, f_{m+1} \in \varpi^{n-d} ind_{KZ}^G \underline{\rho}_k^0$. We will show that $f_{m-1} \in \varpi^{n-d} ind_{KZ}^G \underline{\rho}_k^0$.

By the above equations, we immediately obtain from (11) (note that $af_m \in \varpi^n ind_{KZ}^G \underline{\rho}_k^0$, since $v_F(a) \geq d$)

$$\sum_{i=j}^{k} \binom{i}{j} c_{i,\mu}^{m-1} [\lambda]^{i-j} \in \varpi^{n-d-j} \mathcal{O}_C \tag{17}$$

for all $\mu \in I_{m-1}$, all $\lambda \in \mathbb{F}_q$, and all $0 \leq j \leq d$.
By Lemma 4.12, it follows that for all i, $c_{i,\mu}^{m-1} \in \varpi^{n-2d}\mathcal{O}_C$.
Next, for any $1 \leq j \leq d$, consider the formulas for $C_{j+l(q-1),\mu}^m$ for any $1 \leq l \leq d$.
Note that $j + l(q-1) \leq d + d(q-1) = dq \leq k$.
Since $k \leq q^2/2$, one has

$$d = \left[\frac{k}{q}\right] \leq \frac{k}{q} \leq \frac{q}{2} \Rightarrow 2d \leq q$$

so that $n - 2d + q \geq n$.
Therefore, we get that

$$\varpi^{j+l(q-1)} \binom{i}{j+l(q-1)} c_{i,\mu}^{m-1} \in \varpi^q c_{i,\mu}^{m-1} \mathcal{O}_C \subseteq \varpi^{n-2d+q}\mathcal{O}_C \subseteq \varpi^n \mathcal{O}_C$$

for all j, l. Since for $i \leq k - d$, $\varpi^d \mid \varpi^{k-i}$ and $c_{i,\mu+\varpi^m[\lambda]}^{m+1} \in \varpi^{n-d}\mathcal{O}_C$, it follows that

$$C_{j+l(q-1),\mu}^m \equiv \sum_{i=k-d+1}^{k} \varpi^{k-i} \binom{i}{j+l(q-1)} \sum_{\lambda \in \kappa_F} c_{i,\mu+\varpi^m[\lambda]}^{m+1} [\lambda]^{i-j-l(q-1)} \equiv 0 \mod \varpi^n \mathcal{O}_C$$

Since $k = d \cdot q + r$, with $r \geq d$, we see that $k - d + 1 - d \cdot q = r + 1 - d \geq 1$, showing that for any $1 \leq l \leq d$, any $k-d+1 \leq i \leq k$, we get $i - j - l(q-1) \geq 1$, hence for any $\lambda \in \kappa_F$, $[\lambda]^{i-j-l(q-1)} = [\lambda]^{i-j}$. (Had $i - j - l(q-1)$ been 0, this is violated when $\lambda = 0$!).
By the induction hypothesis, we know that $c_{k-i,\mu}^{m+1} \in \varpi^{n-d}\mathcal{O}_C$. Write, for $0 \leq i \leq d-1$ and $1 \leq j \leq d$, $\sum_{\lambda \in \mathbb{F}_q} c_{k-i,\mu+\pi^m[\lambda]}^{m+1} [\lambda]^{k-i-j} = \varpi^{n-d} \cdot x_{ij}$ for some $x_{ij} \in \mathcal{O}_C$. Then the above equations for $1 \leq l \leq d$ yield

$$\sum_{i=0}^{d-1} \varpi^i \binom{k-i}{j+l(q-1)} \cdot x_{ij} \equiv 0 \mod \varpi^d \tag{18}$$

Let us prove that that $x_{ij} \in \varpi^{j-i}\mathcal{O}_C$ for all $1 \leq j \leq d$ and all $0 \leq i \leq j$. Note that for $i = j$, it is trivial, so we will prove it for $0 \leq i \leq j-1$.
Indeed, fix j. Then, looking modulo ϖ^j, and setting $y_{ij} = \varpi^i x_{ij}$, one obtains the equations (for all $0 \leq i \leq j-1$ and all $1 \leq l \leq j$)

$$\sum_{i=0}^{j-1} \binom{k-i}{j+l(q-1)} \cdot y_{ij} \equiv 0 \mod \varpi^j.$$

By Corollary 4.17, with $m = j$, we see that the matrix of coefficients here is nonsingular modulo p, hence also invertible modulo ϖ^j, and it follows that $y_{ij} \in \varpi^j \mathcal{O}_C$ for all $0 \le i \le j-1$. But this precisely means that

$$x_{ij} = \varpi^{-i} y_{ij} \in \varpi^{j-i} \mathcal{O}_C$$

as claimed.

Therefore,

$$\varpi^i \cdot \sum_{\lambda \in \mathbb{F}_q} c^{m+1}_{k-i,\mu+\varpi^m[\lambda]} \cdot [\lambda]^{k-i-j} = \varpi^i \cdot \varpi^{n-d} x_{ij} \in \varpi^{n-d+j} \mathcal{O}_C$$

Considering now the formulas for $C^m_{j,\mu}$, with $1 \le j \le d$, we get

$$\sum_{i=j}^{k} \binom{i}{j} c^{m-1}_{i,\mu} [\lambda]^{i-j} \in \varpi^{n-d} \mathcal{O}_C.$$

This also holds when $j = 0$ trivially as a conequence of (17).

Hence, applying once more Lemma 4.12,

$$c^{m-1}_{i,\mu} \in \varpi^{n-d} \mathcal{O}_C$$

as claimed. Therefore, in this case, taking $\epsilon = d$ suffices. □

4.4. The case $0 < v_F(a) \le e$

In this subsection, we will prove the following theorem. Since the case $v_F(a) = 0$ is covered by [9, Prop. 4.10], it establishes (i) and (ii) in Theorem 4.6, for that case.

Theorem 4.19. *Let $0 \le k \le q^2/2$. Assume further that $k = dq + r$ with $0 \le r < q - d$. Let $a \in \mathcal{O}_C$ be such that $0 < v_F(a) \le e$. Assume either that $0 < v_F(a) \le 1$ or that $2v_F(a) - 1 \le r$. Then (k, a) is separated.*

We prove the theorem by considering two cases.

We shall first prove the case where $\max(2v_F(a) - 1, 1) \le r$, and then the case $r = 0, v_F(a) \le 1$.

Unfortunately, we have not been able to provide a proof for the case $0 \le r < 2v_F(a) - 1$.

Proof. Let $f \in ind^G_{KZ}\underline{\rho}_k$ be such that $(T - a)f \in B_N + \varpi^n ind^G_{KZ}\underline{\rho}^0_k$. We may assume that $f = \sum_{m=0}^{M} f_m$ where $f_m \in S_{N+m}$, and denote $f_m = 0$ for $m > M$. Looking at S_{N+m}, we have the equations

$$T^-(f_{m+1}) + T^+(f_{m-1}) - af_m \in \varpi^n ind^G_{KZ}\underline{\rho}^0_k \quad 1 \le m \le M+1 \tag{19}$$

Our proof will be based on descending induction on m, showing that if f_m, f_{m+1} are highly divisible, so must be f_{m-1}.

We will initially obtain some bound for the valuation of f_{m-1} using f_m and f_{m+1}, and then we will use that initial bound to bootstrap and obtain better bounds on the valuation of f_m,f_{m+1} and, in turn, f_{m-1}.

Moreover, we may assume that $f_m \in S^0_{N+m}$, using G-equivariance.

We refer the reader to the definition of the coefficients $c^m_{\underline{j},\mu}$ in Corollary 3.13, and to formula (11).

As under our assumptions $|S^+| = 1$, we will usually replace the multi-index notation $\underline{j}$ by $j = j_\sigma$.

The idea of this part of the proof is as follows - the contribution from the T^+ part (the inner vertex) has high valuation when j is large, while the contribution from the T^- part (the outer vertices) has high valuation when j is small.

Let us introduce the statements $\mathscr{A}_m, \mathscr{B}_m, \mathscr{C}_m, \mathscr{D}_m$ for the rest of the proof.

The assumptions $\mathscr{A}_m$ are made to ensure that for small values of j, the contribution from T^+ is of high enough valuation, hence we can infer something about its preimage (by the previous Lemmata). These give us the initial bound for the valuation of f_{m-1}.

In the bootstrapping part, this bound shows that for large values of j, the main contribution comes from T^-, whence we must use $\mathscr{B}_m$ in order to obtain better bounds on the valuation of f_m. These bounds for large values of j can improve our bounds for small values of j by using the assumption $\mathscr{C}_m$, which is a linear relation involving one small value of j, while all the others are large.

Finally, this is used to obtain a better bound on the valuation of f_{m-1}, establishing the theorem.

$$\mathscr{A}_m : \quad c^m_{j,\mu} \in \frac{\varpi^{n-j}}{a} \cdot \mathcal{O}_C \quad \forall 0 \le j \le d, \quad c^m_{i,\mu} \in \frac{\varpi^{n-d}}{a} \cdot \mathcal{O}_C \quad \forall 0 \le i \le k \quad \forall \mu \in I_m$$

$$\mathscr{B}_m : \quad c^m_{k-j,\mu} \in \frac{\varpi^{n-j}}{a} \cdot \mathcal{O}_C \quad \forall 0 \le j \le d, \quad c^m_{i,\mu} \in \frac{\varpi^{n-d}}{a} \cdot \mathcal{O}_C \quad \forall 0 \le i \le k \quad \forall \mu \in I_m$$

$$\mathscr{C}_m : \quad \sum_{s=j}^{\lfloor \frac{k-i}{q-1} \rfloor} \binom{s}{j} \cdot c^m_{i+s(q-1),\mu} \in \frac{\varpi^{n-j}}{a} \cdot \mathcal{O}_C \quad \forall j+1 \le i \le j+q-1, \quad \forall 0 \le j \le d$$

$$\mathscr{D}_m : \quad c^m_{i,\mu} \in \frac{\varpi^n}{a^2} \cdot \mathcal{O}_C \quad \forall 0 \le i \le k$$

Assume, by descending induction on m, that $\mathscr{A}_m, \mathscr{B}_m, \mathscr{B}_{m+1}, \mathscr{C}_m$ hold for all μ, λ.

Note that, as $f_{M+1} = f_{M+2} = 0$, they trivially hold for $m = M+1$. We will prove that $\mathscr{A}_{m-1}, \mathscr{B}_{m-1}, \mathscr{B}_m, \mathscr{C}_{m-1}$ hold.

For this, we make use of the subsequent Lemma 4.20.

We assume $\mathscr{A}_m, \mathscr{B}_m, \mathscr{B}_{m+1}, \mathscr{C}_m$, hence by Lemma 4.20, we know that $\mathscr{A}_{m-1}, \mathscr{C}_{m-1}, \mathscr{D}_m$ also hold.

It remains to show that $\mathscr{B}_{m-1}$ holds. In fact, we need only to show that $c_{k-j,\mu}^{m-1} \in \frac{\varpi^{n-j}}{a}\mathcal{O}_C$ for all $0 \le j \le d$.

Note that since $\mathscr{B}_m$ holds, by applying Lemma 4.20 to $m-1$, we see that $\mathscr{A}_{m-2}, \mathscr{C}_{m-2}$ hold as well, and so does $\mathscr{D}_{m-1}$.

Next, we see from $\mathscr{D}_{m-1}$ that we have $c_{k-j,\mu}^{m-1} \in \frac{\varpi^n}{a^2}\mathcal{O}_C \subseteq \frac{\varpi^{n-j}}{a}\mathcal{O}_C$ for all $v_F(a) \le j \le d$, which we get "for free". Therefore, it remains to show that $c_{k-j,\mu}^{m-1} \in \frac{\varpi^{n-j}}{a} \cdot \mathcal{O}_C$ for all $0 \le j < \min(v_F(a), d)$.

Fix some $0 \le j < \min(v_F(a), d)$.

Now, since by Lemma 4.14, $p \mid \binom{i}{k-j-l(q-1)}$ for all $k - 2v_F(a) < i \le k$ and all $j+1 \le l \le d$ (here we use $2v_F(a)-1 \le r < q-d$), and by $\mathscr{B}_m$, $c_{i,\mu}^m \in \frac{\varpi^{n-k+i}}{a}\mathcal{O}_C$ for all $k - 2v_F(a) < i \le k$, we get (as $\varpi^e \mid p$) that

$$\varpi^{k-i} \cdot \binom{i}{k-j-l(q-1)} \cdot c_{i,\mu}^m \in \varpi^{k-i+e} \cdot \frac{\varpi^{n-k+i}}{a} \cdot \mathcal{O}_C = \frac{\varpi^{n+e}}{a} \cdot \mathcal{O}_C \subseteq \varpi^n \mathcal{O}_C \tag{20}$$

where the last inclusion follows from $v_F(a) \le e$.

Furthermore, since we have shown $\mathscr{D}_m$, we know that $c_{i,\mu}^m \in \frac{\varpi^n}{a^2}\mathcal{O}_C = \varpi^{n-2v_F(a)}\mathcal{O}_C$ for all $0 \le i \le k$, hence for $i \le k - 2v_F(a)$, we get

$$\varpi^{k-i} \cdot c_{i,\mu}^m \in \varpi^{2v_F(a)} \cdot \varpi^{n-2v_F(a)}\mathcal{O}_C = \varpi^n \mathcal{O}_C. \tag{21}$$

At this point we make use of the hypothesis (19).

It then follows from equation (11) for $C_{k-j-l(q-1)}^{m-1}$, and equations (20), (21) that for all $\mu \in I_{m-1}$

$$\varpi^{k-j-l(q-1)} \cdot \sum_{i=k-j-l(q-1)}^{k} \binom{i}{k-j-l(q-1)} \cdot c_{i,[\mu]_{m-2}}^{m-2} [-\lambda_\mu]^{i-k+j+l(q-1)} - \\ -a \cdot c_{k-j-l(q-1),\mu}^{m-1} \in \varpi^n \mathcal{O}_C.$$

But recall that $l \le d$, so that

$$k-j-l(q-1) = (d-l)\cdot(q-1)+(r+d-j) \ge r+d-j \ge d+\max(1, 2v_F(a)-1)-j$$

where in the last inequality we use our assumption that $r \ge 1$.

Since we have established $\mathscr{A}_{m-2}$, we know that $c_{i,\mu}^{m-2} \in \frac{\varpi^{n-d}}{a} \cdot \mathcal{O}_C$, hence $\varpi^{k-j-l(q-1)} \cdot c_{i,\mu}^{m-2} \in \frac{\varpi^{n+\max(1,2v_F(a)-1)-j}}{a} \cdot \mathcal{O}_C \subseteq \varpi^{n-j}\mathcal{O}_C$.

Therefore, we obtain that $a \cdot c_{k-j-l(q-1),\mu}^{m-1} \in \varpi^{n-j}\mathcal{O}_C$, hence

$$c_{k-j-l(q-1),\mu}^{m-1} \in \frac{\varpi^{n-j}}{a}\mathcal{O}_C \quad \forall j+1 \le l \le d. \tag{22}$$

We shall now use $\mathscr{C}_{m-1}$ to infer from the divisibility of these coefficients, the divisibility of the coefficient $c_{k-j,\mu}^{m-1}$ by $\frac{\varpi^{n-j}}{a}$ as desired. This shall be done as follows.

Let i be the unique integer satisfying $j+1 \le i \le j+q-1$ such that $i \equiv k-j \mod (q-1)$, and let $l_0 = \left\lfloor \frac{k-i}{q-1} \right\rfloor$, so that $k-j = i + l_0(q-1)$. (Recall that $k-j+q-1 \ge k-d+q-1 > k$).

If $i < q$, we let $A \in \mathbb{Z}^{(j+1)\times(j+1)}$ be the matrix with entries $A_{tl} = \binom{l_0-l}{t}_{t,l=0}^{j}$.

If $i \ge q$, we let A be the matrix with entries

$$A_{tl} = \begin{cases} \binom{l_0-l}{t} & i-q < t \le j \\ \binom{l_0-l+1}{t} & 0 \le t \le i-q \end{cases} \quad \forall l \in \{0,1,2,\dots,j\}$$

In each of the cases, $A \in GL_{j+1}(\mathbb{Z})$, either by Lemma 4.15 or by Corollary 4.16. Therefore, there exists a non-trivial $\mathbb{Z}$-linear combination of its rows, some $\alpha_t \in \mathbb{Z}$ such that for all $0 \le l \le j$

$$\sum_{t=0}^{j} \alpha_t A_{tl} = \delta_{l,0}. \tag{23}$$

For $t > i-q$, substituting in $\mathscr{C}_{m-1}$ the value t for j, we obtain for all $\mu \in I_{m-1}$

$$\Xi_t := \sum_{s=t}^{l_0} \binom{s}{t} \cdot c^{m-1}_{i+s(q-1),\mu} \in \frac{\varpi^{n-t}}{a} \cdot \mathcal{O}_C \subseteq \frac{\varpi^{n-j}}{a} \cdot \mathcal{O}_C.$$

Note that indeed $t+1 \le j+1 \le i \le t+q-1$, as required.

For $0 \le t \le i-q$, substituting in $\mathscr{C}_{m-1}$ the value t for j and the value $i-(q-1)$ for i, we obtain for all $\mu \in I_{m-1}$

$$\Xi_t := \sum_{s=t-1}^{l_0} \binom{s+1}{t} \cdot c^{m-1}_{i+s(q-1),\mu} = \sum_{s=t}^{l_0} \binom{s}{t} \cdot c^{m-1}_{i+(s-1)(q-1),\mu} \in \frac{\varpi^{n-t}}{a} \cdot \mathcal{O}_C \subseteq \frac{\varpi^{n-j}}{a} \cdot \mathcal{O}_C.$$

Note that indeed $t+1 \le i-(q-1) \le j \le d-1 \le q-1 \le t+q-1$, as required.

Considering the linear combination $\sum_{t=0}^{j} \alpha_t \Xi_t$, we see that

$$\sum_{t=0}^{i-q} \sum_{s=t-1}^{l_0} \alpha_t \binom{s+1}{t} \cdot c^{m-1}_{i+s(q-1),\mu} + \sum_{t=i-q+1}^{j} \sum_{s=t}^{l_0} \alpha_t \binom{s}{t} \cdot c^{m-1}_{i+s(q-1),\mu} =$$

$$= \sum_{t=0}^{j} \alpha_t \Xi_t \in \frac{\varpi^{n-j}}{a} \cdot \mathcal{O}_C$$

which, reindexing, is the same as

$$\sum_{l=0}^{l_0+1} \left(\sum_{t=0}^{i-q} \alpha_t \binom{l_0-l+1}{t} \right) \cdot c^{m-1}_{k-j-l(q-1),\mu} + \sum_{l=0}^{l_0} \left(\sum_{t=i-q+1}^{j} \alpha_t \binom{l_0-l}{t} \right) \cdot c^{m-1}_{k-j-l(q-1),\mu} = \tag{24}$$

$$= \sum_{l=0}^{l_0+1}\left(\sum_{t=0}^{i-q}\alpha_t\binom{l_0-l+1}{t}\right)\cdot c^{m-1}_{i+l_0(q-1)-l(q-1),\mu}+\sum_{l=0}^{l_0}\left(\sum_{t=i-q+1}^{j}\alpha_t\binom{l_0-l}{t}\right)\cdot c^{m-1}_{i+l_0(q-1)-l(q-1),\mu}$$

which lies in $\frac{\varpi^{n-j}}{a}\cdot\mathcal{O}_C$.

Since we assumed that $r < q-d$ we have

$$k-j-(d+1)(q-1) \le k-(d+1)(q-1) = d\cdot q + r - (dq+q-d-1) =$$

$$= r-(q-d-1) \le 0 < 1 \le j+1 \le i = k-j-l_0(q-1)$$

showing that $l_0 \le d$, hence for every $j+1 \le l \le l_0$, by (22) we have $c^{m-1}_{k-j-l(q-1),\mu} \in \frac{\varpi^{n-j}}{a}\mathcal{O}_C$, so that (24) yields

$$\sum_{l=0}^{j}\left(\sum_{t=0}^{j}\alpha_t A_{tl}\right)\cdot c^{m-1}_{k-j-l(q-1),\mu} =$$

$$= \sum_{l=0}^{j}\left(\sum_{t=0}^{i-q}\alpha_t\binom{l_0-l+1}{t}+\sum_{t=i-q+1}^{j}\alpha_t\binom{l_0-l}{t}\right)\cdot c^{m-1}_{k-j-l(q-1),\mu} \in \frac{\varpi^{n-j}}{a}\cdot\mathcal{O}_C.$$

Now we apply (23) to see that this is no more than $c^{m-1}_{k-j,\mu} \in \frac{\varpi^{n-j}}{a}\cdot\mathcal{O}_C$, as wanted. This establishes $\mathscr{B}_{m-1}$.

At this point, we have established $\mathscr{A}_{m-1},\mathscr{B}_{m-1},\mathscr{B}_m,\mathscr{C}_{m-1}$ from $\mathscr{A}_m,\mathscr{B}_m,\mathscr{B}_{m+1},\mathscr{C}_m$. By descending induction, this shows that $\mathscr{A}_m,\mathscr{B}_m,\mathscr{B}_{m+1},\mathscr{C}_m$ hold for all m.

In particular, considering for example $\mathscr{A}_m$, we see that for any m, $c^m_{i,\mu} \in \frac{\varpi^{n-d}}{a}\cdot\mathcal{O}_C$ for all $0 \le i \le k$, showing that $f_m \in \varpi^{n-(d+v_F(a))}\cdot\mathcal{O}_C$ for all m.

Thus, we have shown that if $(T-a)f \in B_N + \varpi^n ind^G_{KZ}\underline{\rho}^0_k$, then $f \in B_{N-1} + \varpi^{n-(d+v_F(a))}\cdot\mathcal{O}_C$.

Therefore, in the case $\max(2v_F(a)-1,1) \le r$, taking $\epsilon = d+v_F(a)$ suffices in order to show that (k,a) is separated. □

Lemma 4.20. *Assume that for some m, $\mathscr{A}_m,\mathscr{B}_{m+1},\mathscr{C}_m$ hold. Then $\mathscr{A}_{m-1},\mathscr{C}_{m-1},\mathscr{D}_m$ hold as well.*

Proof. From (19) and (11) we see that for any $0 \le j \le d$

$$C^m_{j,\mu} = \sum_{i=j}^{k}\varpi^{k-i}\binom{i}{j}\sum_{\lambda\in\kappa_F}c^{m+1}_{i,\mu+\varpi^m[\lambda]}[\lambda]^{i-j}+\varpi^j\sum_{i=j}^{k}c^{m-1}_{i,[\mu]_{m-1}}\binom{i}{j}(-\lambda_\mu)^{i-j}-ac^m_{j,\mu} \in \varpi^n\mathcal{O}_C \tag{25}$$

where $\lambda_\mu = \frac{\mu-[\mu]_{m-1}}{\varpi^{m-1}}$.

By the hypothesis $\mathscr{B}_{m+1}$, for any $k-d < i \le k$ (and any μ), we have $c^{m+1}_{i,\mu} \in \frac{\varpi^{n-k+i}}{a}\cdot\mathcal{O}_C$, hence $\varpi^{k-i}\cdot c^{m+1}_{i,\mu} \in \frac{\varpi^n}{a}\cdot\mathcal{O}_C$.

Also, for any $0 \le i \le k-d$, by $\mathscr{B}_{m+1}$, we have $c_{i,\mu}^{m+1} \in \frac{\varpi^{n-d}}{a} \cdot \mathcal{O}_C$, hence $\varpi^{k-i} \cdot c_{i,\mu}^{m+1} \in \frac{\varpi^d \cdot \varpi^{n-d}}{a} \cdot \mathcal{O}_C = \frac{\varpi^n}{a} \cdot \mathcal{O}_C$.
We conclude that for any $0 \le i \le k$, one has

$$\varpi^{k-i} \cdot c_{i,\mu}^{m+1} \in \frac{\varpi^n}{a} \cdot \mathcal{O}_C. \tag{26}$$

This implies that the first sum in (25) lies in $\frac{\varpi^n}{a} \cdot \mathcal{O}_C$, hence

$$\varpi^j \sum_{i=j}^{k} c_{i,[\mu]_{m-1}}^{m-1} \binom{i}{j} (-\lambda_\mu)^{i-j} - a c_{j,\mu}^m \in \frac{\varpi^n}{a} \mathcal{O}_C \tag{27}$$

Furthermore, for any $0 \le j \le d$, by $\mathscr{A}_m$, we know that $c_{j,\mu}^m \in \frac{\varpi^{n-j}}{a} \cdot \mathcal{O}_C$, hence

$$a c_{j,\mu}^m \in \varpi^{n-j} \mathcal{O}_C \tag{28}$$

• If $v_F(a) \le j$, we see that $\varpi^n / a \in \varpi^{n-j} \mathcal{O}_C$, so we get from (26), (28), and (25) that

$$\varpi^j \sum_{i=j}^{k} \binom{i}{j} c_{i,\mu}^{m-1} [\lambda]^{i-j} \in \varpi^{n-j} \mathcal{O}_C \Rightarrow \sum_{i=j}^{k} \binom{i}{j} c_{i,\mu}^{m-1} [\lambda]^{i-j} \in \varpi^{n-2j} \mathcal{O}_C \tag{29}$$

for all $v_F(a) \le j \le d$, for all $\mu \in I_{m-1}$ and for all $\lambda \in \kappa_F$.
• If $j \le v_F(a)$, we see that $\varpi^{n-j} \in \frac{\varpi^n}{a} \cdot \mathcal{O}_C$, so we get from (26), (28), and (25) that

$$\varpi^j \sum_{i=j}^{k} \binom{i}{j} c_{i,\mu}^{m-1} [\lambda]^{i-j} \in \frac{\varpi^n}{a} \cdot \mathcal{O}_C \Rightarrow \sum_{i=j}^{k} \binom{i}{j} c_{i,\mu}^{m-1} [\lambda]^{i-j} \in \frac{\varpi^{n-j}}{a} \cdot \mathcal{O}_C. \tag{30}$$

In particular, by Lemma 4.12, we see that if $v_F(a) \le d$, then $c_{i,\mu}^{m-1} \in \varpi^{n-2d} \mathcal{O}_C$ for all $0 \le i \le k$, and if $v_F(a) \ge d$, then $c_{i,\mu}^{m-1} \in \frac{\varpi^{n-d}}{a} \cdot \mathcal{O}_C$ for all $0 \le i \le k$.
Substituting $\lambda = 0$ in (30) we get $c_{j,\mu}^{m-1} \in \frac{\varpi^{n-j}}{a} \mathcal{O}_C$.
Therefore, if $v_F(a) \ge d$, we have already established $\mathscr{A}_{m-1}$. In this case, since $\frac{\varpi^{n-d}}{a} \in \frac{\varpi^n}{a^2} \cdot \mathcal{O}_C$, $\mathscr{D}_m$ trivially holds.
If $v_F(a) < d$, we consider the coefficients $C_{2d,\mu}^m, C_{2d+1,\mu}^m, \ldots, C_{k,\mu}^m$. By (25) and (26), using the fact that $c_{i,\mu}^{m-1} \in \varpi^{n-2d} \mathcal{O}_C$ for all i, we get that $a c_{j,\mu}^m \in \frac{\varpi^n}{a} \mathcal{O}_C$ for all $j \ge 2d$.
In particular, since, by assumption, $q \ge 2k/q \ge 2d$, we get that for any $1 \le j \le 2d-1$ and any $1 \le l$,

$$j + l(q-1) \ge 1 + (q-1) = q \ge 2d$$

hence $c_{j+l(q-1),\mu}^m \in \frac{\varpi^n}{a^2} \mathcal{O}_C$.

By the assumption $\mathscr{C}_m$ (substituting j for i and 0 for j), it follows also that $c^m_{j,\mu} \in \frac{\varpi^n}{a^2}\mathcal{O}_C$. Therefore $ac^m_{j,\mu} \in \frac{\varpi^n}{a}\mathcal{O}_C$ for all $1 \leq j \leq 2d-1$, hence for all $0 \leq j \leq k$, establishing $\mathscr{D}_m$. Note that the case $j = 0$ is given by $\mathscr{A}_m$.

We may now consider once more the equations for $C^m_{1,\mu}, \ldots, C^m_{d,\mu}$, and get from (26), (25) and $\mathscr{D}_m$ that $\forall 1 \leq j \leq d$

$$\varpi^j \sum_{i=j}^{k} \binom{i}{j} c^{m-1}_{i,\mu}[\lambda]^{i-j} \in \frac{\varpi^n}{a}\mathcal{O}_C \Rightarrow \sum_{i=j}^{k} \binom{i}{j} c^{m-1}_{i,\mu}[\lambda]^{i-j} \in \frac{\varpi^{n-j}}{a}\mathcal{O}_C.$$

When $j = 0$, this holds by (30). By Lemma 4.12, it follows that $c^{m-1}_{i,\mu} \in \frac{\varpi^{n-d}}{a}\mathcal{O}_C$ for all i. Also, it shows that $c^{m-1}_{j,\mu} \in \frac{\varpi^{n-j}}{a}\mathcal{O}_C$ for $1 \leq j \leq d$, by substituting $\lambda = 0$. Therefore, we have established $\mathscr{A}_{m-1}$ in this case as well.

Finally, for any $0 \leq j \leq d$, and for any $0 \leq t \leq j$

$$\sum_{i=t}^{k} \binom{i}{t} c^{m-1}_{i,\mu}[\lambda]^{i-t} \in \frac{\varpi^{n-t}}{a} \cdot \mathcal{O}_C \subseteq \frac{\varpi^{n-j}}{a} \cdot \mathcal{O}_C$$

for all $\lambda \in \kappa_F$. Thus, by Lemma 4.13, substituting j for d and i for j, we get $\mathscr{C}_{m-1}$. □

We now consider the case $0 < v_F(a) \leq 1$ and $r = 0$, using a different argument.

Theorem 4.21. *Let $k = dq$, and assume $1 \leq d < \frac{q}{2}$ (note that this excludes $q = 2$) . Let $a \in \mathcal{O}_C$ be such that $0 < v_F(a) \leq 1$, and let $N \in \mathbb{Z}_{>0}$. There exists a constant $\epsilon \in \mathbb{Z}_{\geq 0}$ depending only on N, k, a such that for all $n \in \mathbb{Z}_{\geq 0}$, and all $f \in ind^G_{KZ}\underline{\rho}^0_k$:*

$$(T-a)(f) \in B_N + \varpi^n ind^G_{KZ}\underline{\rho}^0_k \Rightarrow f \in B_{N-1} + \varpi^{n-\epsilon} ind^G_{KZ}\underline{\rho}^0_k$$

Proof. We may assume that $f = \sum_{m=0}^{M} f_m$ where $f_m \in S^0_{N+m}$, and denote $f_m = 0$ for $m > M$. Looking at S_{N+m} , we have the equations

$$T^-(f_{m+1}) + T^+(f_{m-1}) - af_m \in \varpi^n ind^G_{KZ}\underline{\rho}^0_k \quad 1 \leq m \leq M+1$$

Assume, by descending induction on m, that the following hold:

$$c^{m+1}_{k,\mu} \in \frac{\varpi^n}{a^2}\mathcal{O}_C, \quad c^{m+1}_{k-j,\mu} \in \frac{\varpi^n}{a}\mathcal{O}_C \quad \forall 0 < j \leq d, \quad c^{m+1}_{i,\mu} \in \frac{\varpi^{n-d}}{a}\mathcal{O}_C \quad \forall 0 \leq i \leq k \quad \forall \mu \in I_{m+1}$$

$$c^m_{0,\mu} \in \frac{\varpi^n}{a}\mathcal{O}_C, \quad \sum_{l=0}^{\lfloor \frac{k-j}{q-1} \rfloor} c^m_{j+l(q-1),\mu} \in \frac{\varpi^n}{a}\mathcal{O}_C \quad \forall 1 \leq j \leq d, \quad c^m_{i,\mu} \in \frac{\varpi^{n-d}}{a}\mathcal{O}_C \quad \forall 0 \leq i \leq k \quad \forall \mu \in I_m$$

$$\sum_{\lambda \in \kappa_F} c^{m+1}_{k,\mu+\varpi^m[\lambda]}[\lambda]^l \in \frac{\varpi^n}{a}\mathcal{O}_C, \quad \forall l \in \{0, 1, 2, \ldots, d, q-1\} \quad \forall \mu \in I_m \qquad (31)$$

We will show that the same formulas hold for $m-1$, hence establish that they hold for all $0 \leq m \leq M+1$.

First, for $\mu \in I_{m-1}$ and $\lambda \in \kappa_F$, consider the formula for $C^m_{0,\mu+\varpi^{m-1}[\lambda]}$, see (11). By (31) with $l = d$, using the fact that $[\lambda]^q = [\lambda]$ for all $\lambda \in \kappa_F$, we know that

$$\sum_{\lambda' \in \kappa_F} c^{m+1}_{k,\mu+\varpi^{m-1}[\lambda]+\varpi^m[\lambda']}[\lambda']^{dq} = \sum_{\lambda' \in \kappa_F} c^{m+1}_{k,\mu+\varpi^{m-1}[\lambda]+\varpi^m[\lambda']}[\lambda']^{d} \in \frac{\varpi^n}{a}\mathcal{O}_C$$

which is the first summand in the first sum in (11) with $j = 0$.

For $i \leq k-d$, since we assume $c^{m+1}_{i,\mu+\varpi^{m-1}[\lambda]+\varpi^m[\lambda']} \in \frac{\varpi^{n-d}}{a}\mathcal{O}_C$, we see that

$$\varpi^{k-i} \cdot \sum_{\lambda' \in \kappa_F} c^{m+1}_{i,\mu+\varpi^{m-1}[\lambda]+\varpi^m[\lambda']}[\lambda']^{i} \in \varpi^d \cdot \frac{\varpi^{n-d}}{a} \cdot \mathcal{O}_C = \frac{\varpi^n}{a} \cdot \mathcal{O}_C$$

Also, for $k-d < i < k$, since we assume $c^{m+1}_{i,\mu+\varpi^{m-1}[\lambda]+\varpi^m[\lambda']} \in \frac{\varpi^n}{a}\mathcal{O}_C$, we get

$$\varpi^{k-i} \cdot \sum_{\lambda' \in \kappa_F} c^{m+1}_{i,\mu+\varpi^{m-1}[\lambda]+\varpi^m[\lambda']}[\lambda']^{i} \in \frac{\varpi^n}{a} \cdot \mathcal{O}_C$$

This shows that the entire first sum in (11) with $j = 0$ lies in $\frac{\varpi^n}{a} \cdot \mathcal{O}_C$. In addition, we have assumed that $c^m_{0,\mu+\varpi^{m-1}[\lambda]} \in \frac{\varpi^n}{a} \cdot \mathcal{O}_C$. Therefore

$$\sum_{i=0}^{k} c^{m-1}_{i,\mu}[\lambda]^i \in \frac{\varpi^n}{a}\mathcal{O}_C$$

Next, we consider the formulas for $C^m_{j,\mu+\varpi^{m-1}[\lambda]}$ with $1 \leq j \leq d$. By (31) with $l = d-j$ for $j \neq d$ and $l = q-1$ for $j = d$, using the fact that $[\lambda]^q = [\lambda]$ for all $\lambda \in \kappa_F$, we know that

$$\binom{k}{j} \sum_{\lambda' \in \kappa_F} c^{m+1}_{k,\mu+\varpi^{m-1}[\lambda]+\varpi^m[\lambda']}[\lambda']^{dq-j} =$$

$$= \binom{k}{j} \sum_{\lambda' \in \kappa_F} c^{m+1}_{k,\mu+\varpi^{m-1}[\lambda]+\varpi^m[\lambda']}[\lambda']^{l} \in \frac{\varpi^n}{a}\mathcal{O}_C \subseteq \varpi^{n-d}\mathcal{O}_C$$

which is the first summand in the first sum in (11).

Since for all i, we have $c^{m+1}_{i,\mu+\varpi^{m-1}[\lambda]+\varpi^m[\lambda']} \in \frac{\varpi^{n-d}}{a}\mathcal{O}_C$, when considering $i < k$ we also have $1 \leq k-i$, hence

$$\varpi^{k-i}\binom{i}{j} \sum_{\lambda' \in \kappa_F} c^{m+1}_{i,\mu+\varpi^{m-1}[\lambda]+\varpi^m[\lambda']}[\lambda']^{i-j} \in \varpi \cdot \frac{\varpi^{n-d}}{a}\mathcal{O}_C \subseteq \varpi^{n-d}\mathcal{O}_C$$

where the last inclusion holds as $v_F(a) \leq 1$. This shows that the entire first sum in (11) lies in $\varpi^{n-d}\mathcal{O}_C$.

Since we also have $c^m_{j,\mu+\varpi^{m-1}[\lambda]} \in \frac{\varpi^{n-d}}{a}\mathcal{O}_C$, by (11) we see that for all $1 \leq j \leq d$

$$\sum_{i=j}^{k} \binom{i}{j} c^{m-1}_{i,\mu}[\lambda]^{i-j} \in \varpi^{n-d-j}\mathcal{O}_C$$

Therefore, by lemma 4.12 we have $c^{m-1}_{i,\mu} \in \varpi^{n-2d}\mathcal{O}_C$ for all i.

Let $0 < j \leq d$. Looking at the formula for $C^m_{k-j,\mu}$, using the fact that $k - j \geq dq - d = d(q-1) \geq 2d$ (recall $q \neq 2$), we see that the second sum satisfies

$$\varpi^{k-j} \sum_{i=k-j}^{k} \binom{i}{k-j} c^{m-1}_{i,[\mu]_{m-1}}[\lambda_\mu]^{i-(k-j)} \in \varpi^{2d} \cdot \varpi^{n-2d}\mathcal{O}_C = \varpi^n\mathcal{O}_C$$

Also, we deduce from the hypothesis (31) with $l = j$ that

$$\binom{k}{k-j} \sum_{\lambda\in\kappa_F} c^{m+1}_{k,\mu+\varpi^m[\lambda]}[\lambda]^{k-(k-j)} =$$

$$= \binom{k}{k-j} \sum_{\lambda\in\kappa_F} c^{m+1}_{k,\mu+\varpi^m[\lambda]}[\lambda]^{j} \in \binom{k}{k-j} \cdot \frac{\varpi^n}{a}\mathcal{O}_C \subseteq \varpi^n\mathcal{O}_C$$

since $p \mid \binom{k}{k-j} = \binom{dq}{(d-1)q+(q-j)}$ by Kummer's theorem, and $v_F(a) \leq 1$.

For $i < k - d$, since we assume $c^{m+1}_{i,\mu+\varpi^m[\lambda]} \in \frac{\varpi^{n-d}}{a}\mathcal{O}_C$, we see that

$$\varpi^{k-i}\binom{i}{j} \cdot \sum_{\lambda\in\kappa_F} c^{m+1}_{i,\mu+\varpi^m[\lambda]}[\lambda]^{i-j} \in \varpi^{d+1} \cdot \frac{\varpi^{n-d}}{a} \cdot \mathcal{O}_C = \frac{\varpi^{n+1}}{a} \cdot \mathcal{O}_C \subseteq \varpi^n\mathcal{O}_C$$

Also, for $k - d \leq i < k$, since we assume $c^{m+1}_{i,\mu+\varpi^m[\lambda]} \in \frac{\varpi^n}{a}\mathcal{O}_C$, and $1 \leq k - i$, we get

$$\varpi^{k-i}\binom{i}{j} \cdot \sum_{\lambda\in\kappa_F} c^{m+1}_{i,\mu+\varpi^m[\lambda]}[\lambda]^{i-j} \in \varpi \cdot \frac{\varpi^n}{a} \cdot \mathcal{O}_C = \varpi^n\mathcal{O}_C$$

This shows that both sums in (11) lie in $\varpi^n\mathcal{O}_C$, hence also

$$a \cdot c^m_{k-j,\mu} \in \varpi^n\mathcal{O}_C$$

Furthermore, for any $1 \leq j \leq d$, looking at the formulas for $C^m_{j+(q-1),\mu}, \ldots, C^m_{j+l(q-1),\mu}, \ldots$, as $j + l(q-1) \geq j + q - 1 \geq q > 2d$, by the same reasoning, we deduce from the hypothesis (31) with $l = d - j$ that

$$a \cdot c^m_{j+l(q-1),\mu} \in \frac{\varpi^n}{a}\mathcal{O}_C$$

Since $\sum_{l=0}^{\lfloor \frac{k-j}{q-1} \rfloor} c_{j+l(q-1),\mu}^m \in \frac{\varpi^n}{a}\mathcal{O}_C$ for $1 \leq j \leq d$, this shows that $a \cdot c_{j,\mu}^m \in \frac{\varpi^n}{a}\mathcal{O}_C$ for all $0 \leq j \leq k$.

We also note that $p \mid \binom{dq}{d+l(q-1)}$ for all $1 \leq l < d$, by Kummer's theorem, therefore showing that

$$a \cdot c_{d+l(q-1),\mu}^m \in \varpi^n \mathcal{O}_C$$

Since $\sum_{l=0}^{d} c_{d+l(q-1),\mu}^m \in \frac{\varpi^n}{a}\mathcal{O}_C$, we deduce that

$$c_{d,\mu}^m + c_{dq,\mu}^m \in \frac{\varpi^n}{a}\mathcal{O}_C \tag{32}$$

Therefore, we have established that

$$c_{k,\mu}^m \in \frac{\varpi^n}{a^2}\mathcal{O}_C, \quad c_{k-j,\mu}^m \in \frac{\varpi^n}{a}\mathcal{O}_C \quad \forall 0 < j \leq d,, \quad c_{i,\mu}^m \in \frac{\varpi^{n-d}}{a}\mathcal{O}_C \quad \forall 0 \leq i \leq k$$

Returning to the formulas for $C_{0,\mu}^m, C_{1,\mu}^m, \ldots, C_{d,\mu}^m$, we see that for all $\lambda \in \mathbb{F}_q$ one has

$$\sum_{i=0}^{k} c_{i,\mu}^{m-1}[\lambda]^i \in \frac{\varpi^n}{a}\mathcal{O}_C, \quad \sum_{i=1}^{k} i c_{i,\mu}^{m-1}[\lambda]^{i-1} \in \frac{\varpi^{n-1}}{a}\mathcal{O}_C, \ldots, \quad \sum_{i=d}^{k} \binom{i}{d} c_{i,\mu}^{m-1}[\lambda]^{i-d} \in \frac{\varpi^{n-d}}{a}\mathcal{O}_C$$

Therefore, by Lemma 4.12 we have $c_{i,\mu}^{m-1} \in \frac{\varpi^{n-d}}{a}\mathcal{O}_C$ for all i. Moreover, we see that

$$c_{0,\mu}^{m-1} \in \frac{\varpi^n}{a}\mathcal{O}_C, \quad \sum_{l=0}^{\lfloor \frac{k-j}{q-1} \rfloor} c_{j+l(q-1),\mu}^m \in \frac{\varpi^n}{a}\mathcal{O}_C \quad \forall 1 \leq j \leq d, \quad c_{i,\mu}^{m-1} \in \frac{\varpi^{n-d}}{a}\mathcal{O}_C \quad \forall 0 \leq i \leq k$$

It remains to establish (31) for m. Looking at the equation for $C_{d,\mu}^m$, we see that for all μ we have

$$\varpi^d \cdot \sum_{i=d}^{k} \binom{i}{d} c_{i,[\mu]_{m-1}}^{m-1}[\lambda_\mu]^{i-d} - a \cdot c_{d,\mu}^m \in \varpi^n \mathcal{O}_C$$

since $p \mid \binom{k}{d} = \binom{dq}{d}$. Fixing $\mu \in I_{m-1}$ and summing over all $\lambda \in \mathbb{F}_q$, we get

$$a \cdot \sum_{\lambda \in \mathbb{F}_q} c_{d,\mu+\varpi^{m-1}[\lambda]}^m [\lambda]^l - \varpi^d \cdot \sum_{i=d}^{k} \binom{i}{d} c_{i,\mu}^{m-1} \sum_{\lambda \in \mathbb{F}_q} [\lambda]^{i+l-d} \in \varpi^n \mathcal{O}_C$$

for any $l \in \{0, 1, 2, \ldots, d, q-1\}$.

However, as

$$\sum_{\lambda \in \mathbb{F}_q} [\lambda]^i \equiv \begin{cases} -1 & q-1 \mid i, \qquad i \neq 0 \\ 0 & else \end{cases} \mod p$$

we obtain

$$a \cdot \sum_{\lambda \in \mathbb{F}_q} c^m_{d,\mu+\varpi^{m-1}[\lambda]}[\lambda]^l - \varpi^d \cdot \sum_{h=1}^{\left[\frac{k-d+l}{q-1}\right]} \binom{d-l+h(q-1)}{d} c^{m-1}_{d-l+h(q-1),\mu} \in \varpi^n \mathcal{O}_C$$

Fix some $l \in \{0, 1, \ldots, d\}$. Note that for all $h \leq d - l$, one has

$$p \mid \binom{d-l+h(q-1)}{d} = \binom{h \cdot q + (d-l-h)}{d}$$

This means we have

$$a \cdot \sum_{\lambda \in \mathbb{F}_q} c^m_{d,\mu+\varpi^{m-1}[\lambda]}[\lambda]^l - \varpi^d \cdot \sum_{h=d-l+1}^{d} \binom{d-l+h(q-1)}{d} c^{m-1}_{d-l+h(q-1),\mu} \in \varpi^n \mathcal{O}_C \tag{33}$$

For $l = 0$, this already implies

$$\sum_{\lambda \in \mathbb{F}_q} c^m_{d,\mu+\varpi^{m-1}[\lambda]} \in \frac{\varpi^n}{a} \mathcal{O}_C$$

hence by (32)

$$\sum_{\lambda \in \mathbb{F}_q} c^m_{dq,\mu+\varpi^{m-1}[\lambda]} \in \frac{\varpi^n}{a} \mathcal{O}_C$$

For arbitary l, we proceed as follows.

Consider the formulas for $C^{m-1}_{0,\mu}, C^{m-1}_{1,\mu}, \ldots, C^{m-1}_{d,\mu}$. We obtain as before that $c^{m-2}_{i,\mu} \in \varpi^{n-2d} \mathcal{O}_C$ for all i.

We may now consider the formulas for $C^{m-1}_{dq-l,\mu}, C^{m-1}_{(d-1)q-l+1,\mu}, \ldots, C^{m-1}_{(d-l+1)q-1,\mu}$. Since

$$(d-l+1)q - 1 \geq q - 1 \geq 2d$$

we get

$$\binom{dq}{d-l+h(q-1)} \cdot \sum_{\lambda \in \mathbb{F}_q} c^m_{dq,\mu+\varpi^{m-1}[\lambda]}[\lambda]^l + a \cdot c^{m-1}_{d-l+h(q-1),\mu} \in \varpi^n \mathcal{O}_C$$

for all $d - l + 1 \leq h \leq d$. Substituing back in (33), we get

$$\left(a + \frac{1}{a} \cdot \sum_{h=d-l+1}^{d} \varpi^d \cdot \binom{dq}{d-l+h(q-1)} \cdot \binom{d-l+h(q-1)}{d}\right) \cdot \sum_{\lambda \in \mathbb{F}_q} c^m_{dq,\mu+\varpi^m[\lambda]}[\lambda]^l \in \varpi^n \mathcal{O}_C$$

Note that $p \mid \binom{dq}{d-l+h(q-1)} = \binom{dq}{(h-1)q+q+d-l-h}$, hence

$$v_F\left(\frac{\varpi^d}{a} \cdot \binom{dq}{d-l+h(q-1)} \cdot \binom{d-l+h(q-1)}{d}\right) \geq d + 1 - v_F(a)$$

But, as $v_F(a) \leq 1 < \frac{1+d}{2}$, it follows that $v_F(a) < d+1-v_F(a)$, so that we must have

$$a \cdot \sum_{\lambda \in \mathbb{F}_q} c^m_{dq,\mu+\varpi^m[\lambda]}[\lambda]^l \in \varpi^n \mathcal{O}_C$$

as claimed.

Finally, looking at the formulas for $C^{m-1}_{dq}, \ldots, C^{m-1}_{d+q-1}$, we have

$$\binom{dq}{d+h(q-1)} \cdot \sum_{\lambda \in \mathbb{F}_q} c^m_{dq,\mu+\varpi^{m-1}[\lambda]}[\lambda]^{q-1} + a \cdot c^{m-1}_{d+h(q-1),\mu} \in \varpi^n \mathcal{O}_C$$

for all $1 \leq h \leq d-1$, and

$$\sum_{\lambda \in \mathbb{F}_q} c^m_{dq,\mu+\varpi^{m-1}[\lambda]} + a \cdot c^{m-1}_{dq,\mu} \in \varpi^n \mathcal{O}_C$$

Substituting in (33), and recalling that $\sum_{h=0}^d c^{m-1}_{d+h(q-1)} \in \frac{\varpi^n}{a}\mathcal{O}_C$, we obtain

$$\left(a + \frac{1}{a} \cdot \varpi^d \cdot \sum_{h=0}^{d-1} \binom{d+h(q-1)}{d} \cdot \binom{dq}{d+h(q-1)}\right) \cdot \sum_{\lambda \in \mathbb{F}_q} c^m_{dq,\mu+\varpi^{m-1}[\lambda]}[\lambda]^l \in \varpi^n \mathcal{O}_C$$

since $\varpi^2 \mid \varpi^d \cdot \binom{dq}{d}$.

Since $v_F(a) \leq 1 < \frac{d+1}{2}$, this is only possible if $\sum_{\lambda \in \mathbb{F}_q} c^m_{dq,\mu+\varpi^{m-1}[\lambda]}[\lambda]^{q-1} \in \frac{\varpi^n}{a}\mathcal{O}_C$. Therefore, we are done, and $\epsilon = d + v_F(a)$ suffices. □

[1] *Laure Barthel, Ron Livne, et al. Irreducible modular representations of* GL$_2$ *of a local field.* Duke Mathematical Journal*, 75(2):261–292, 1994.*

[2] *Christophe Breuil. Sur quelques représentations modulaires et p-adiques de $GL_2(\mathbb{Q}_p)$: I.* Compositio Mathematica*, 138(2):165–188, 2003.*

[3] *Christophe Breuil. Sur quelques representations modulaires et p-adiques de $GL_2(\mathbb{Q}_p)$. II.* JOURNAL-INSTITUTE OF MATHEMATICS OF JUSSIEU*, 2(1):23–58, 2003.*

[4] *Christophe Breuil and Peter Schneider. First steps towards p-adic langlands functoriality.* Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2007(610):149–180, 2007.*

[5] *Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paskunas, and Sug Woo Shin. Patching and the p-adic local langlands correspondence.* arXiv preprint arXiv:1310.0831*, 2013.*

[6] *Pierre Colmez. Représentations de $GL_2(\mathbb{Q}_p)$ et (ϕ, Γ)-modules.* Astérisque*, 330(281):509, 2010.*

[7] *Pierre Colmez and Jean-Marc Fontaine. Construction des représentations p-adiques semi-stables.* Inventiones mathematicae*, 140(1):1–43, 2000.*

[8] *Pierre Colmez, Gabriel Dospinescu, and Vytautas Paskunas. The p-adic local langlands correspondence for $GL_2(\mathbb{Q}_p)$.* arXiv preprint arXiv:1310.2235*, 2013.*

[9] *Marco De Ieso. Existence de normes invariantes pour GL_2.* Journal of Number Theory*, 133(8):2729–2755, 2013.*

[10] *Matthew Emerton. Local-global compatibility in the p-adic langlands programme for $GL_2(\mathbb{Q}_p)$.* preprint*, 2010.*

[11] *Matthew Emerton et al. p-adic L-functions and unitary completions of representations of p-adic reductive groups.* Duke Mathematical Journal*, 130(2):353–392, 2005.*

[12] *Yongquan Hu. Normes invariantes et existence de filtrations admissibles.* Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2009 (634):107–141, 2009.*

[13] *Vytautas Paškūnas. The image of colmez montreal functor.* Publications mathématiques de l'IHÉS*, 118(1):1–191, 2013.*

[14] *Peter Schneider and Jeremy Teitelbaum. Locally analytic distributions and p-adic representation theory, with applications to GL_2.* Journal of the American Mathematical Society*, 15(2):443–468, 2002.*

[15] *Peter Schneider and Jeremy Teitelbaum. Algebras of p-adic distributions and admissible representations.* Inventiones mathematicae*, 153(1):145–196, 2003.*

[16] *Peter Schneider, Jeremy Teitelbaum, Das Uferlose, and das Leben Stürmende. Banach-hecke algebras and p-adic galois representations.* Doc. Math*, pages 631–684, 2006.*

[17] *Jean Pierre Serre.* Trees. *Springer, 1980.*

[18] *Claus M Sorensen. A proof of the breuil-schneider conjecture in the indecomposable case.* Annals of Mathematics*, 177:1–16, 2013.*

4 Existence of Invariant Norms in p-adic Representations of $U_3(F)$

Eran Assaf
Unpublished

Existence of Invariant Norms for p-adic representations of $U_3(F)$

Eran Assaf

Abstract

Let p be a prime, and let F be a finite extension of $\mathbb{Q}_p$. The p-adic Langlands programme attempts to associate certain representations of $GL_n(F)$ with certain n-dimensional representations of $Gal(\overline{\mathbb{Q}}_p/F)$. More generally, one expects to associate certain packets of representations of a reductive group G with certain conjugacy classes of homomorphisms $Gal(\overline{\mathbb{Q}}_p/F) \to {}^{\mathrm{L}}G$. If V is a two-dimensional p-adic representation of the group $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, it is known how to associate to it a continuous p-adic representation $B(V)$ of $GL_2(\mathbb{Q}_p)$. If F is a non-trivial finite extension of $\mathbb{Q}_p$, then a way of associating p-adic representations of $GL_2(F)$ to two-dimensional p-adic representations of $Gal(\overline{\mathbb{Q}}_p/F)$ is yet to be found. Such is the case also for $GL_n(F)$ or other reductive groups defined over $\mathbb{Q}_p$. One of the main tools in establishing the correspondence for $GL_2(\mathbb{Q}_p)$ was the existence of $GL_2(\mathbb{Q}_p)$-invariant norms in certain locally algebraic representations of $GL_2(\mathbb{Q}_p)$. We prove criteria for the existence of such norms in certain locally algebraic representations of $U_3(F)$, where F is a finite extension of $\mathbb{Q}_p$. This provides new instances of the Breuil-Schneider conjecture (generalized to quasi-split groups) about the existence of invariant norms on certain locally algebraic representations of reductive groups.

Contents

Preprint submitted to Elsevier *April 14, 2017*

1. Introduction

1.1. *p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$*

The classical local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ is a bijection between certain two-dimensional Weil-Deligne representations of the Weil group $W_{\mathbb{Q}_p}$ and irreducible smooth representations of $GL_2(\mathbb{Q}_p)$.

In this correspondence, the topology of the coefficient field, C, plays no role. A natural source of Weil-Deligne representations are Galois representations.

Thus, for $l \neq p$, with any continuous representation $\rho : \Gamma_{\mathbb{Q}_p} = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(\overline{\mathbb{Q}}_l)$ one can associate a Weil-Deligne representation $WD(\rho)$, and hence, by the classical local Langlands correspondence, an irreducible smooth representation $\pi_{sm}(\rho)$ of $GL_2(\mathbb{Q}_p)$ over $\bar{\mathbb{Q}}_l$. Moreover, one can recover ρ from $\pi_{sm}(\rho)$. This completely fails for $l = p$.

In the case $l = p$, if C is a finite extension of $\mathbb{Q}_p$, and $\rho : \Gamma_{\mathbb{Q}_p} \to GL_2(C)$ is potentially semistable, Fontaine showed that one can still attach a Weil-Deligne representation to ρ, $WD(\rho)$, and hence $\pi_{sm}(\rho)$ still makes sense (see Breuil and Schneider [6]). However, $\rho \rightsquigarrow \pi_{sm}(\rho)$ is no longer reversible. In addition, ρ admits Hodge-Tate weights, which correspond to an irreducible algebraic representation $\pi_{alg}(\rho)$ of $GL_2(\mathbb{Q}_p)$.

Still, one cannot reconstruct ρ from $\pi_{sm}(\rho)$ and $\pi_{alg}(\rho)$. In p-adic Hodge theory, the potentially semistable ρ are classified by linear algebra data which includes a certain Hodge filtration, which is lost in the process of constructing these representations.

The p-adic local Langlands correspondence takes any continuous representation $\rho : \Gamma_{\mathbb{Q}_p} \to GL_2(C)$ and attaches to it a C-Banach space $\Pi(\rho)$ with a unitary $GL_2(\mathbb{Q}_p)$-action.

This map $\rho \rightsquigarrow \Pi(\rho)$ is reversible, and compatible with classical local Langlands in the following sense: When ρ is potentially semistable, with distinct Hodge-Tate weights,

$$\Pi(\rho)^{alg} = \pi_{alg}(\rho) \otimes_C \pi_{sm}(\rho)$$

Furthermore, $\Pi(\rho)^{alg} = 0$ otherwise. Here V^{alg} are the locally algebraic vectors in V, as will be defined in Definition 3.6.

When ρ is irreducible, $\Pi(\rho)$ is known to be topologically irreducible, and therefore the completion of $\pi_{alg}(\rho) \otimes \pi_{sm}(\rho)$ relative to a suitable $GL_2(\mathbb{Q}_p)$-invariant norm, which corresponds to the lost Hodge filtration.

1.2. *The Breuil-Schneider conjecture*

Let F be a finite extension of $\mathbb{Q}_p$, with residue field of cardinality $q = p^f$. The p-adic local Langlands correspondence remains unknown for $GL_2(F)$.

Using the case of $GL_2(\mathbb{Q}_p)$ as a guiding principle, $BS(\rho) := \pi_{alg}(\rho) \otimes_C \pi_{sm}(\rho)$ can be defined for any potentially semi-stable representation

$$\rho : \Gamma_F = Gal(\overline{\mathbb{Q}}_p/F) \to GL_n(C)$$

with distinct Hodge-Tate weights, and the Breuil-Schneider conjecture is that $BS(\rho)$ admits a $GL_n(F)$-invariant norm. The resulting completion with respect to one of these norms should be closely related to the yet undefined $\Pi(\rho)$. Let us state the exact formulation. Assume the Galois closure of F is contained in C.

Let $\mathcal{D} = (WD, HT_\tau)_{\tau \in Hom(F,C)}$ be data consisting of a Weil-Deligne representation WD, and tuples of integers $HT_\tau = \{w_{1,\tau} < \ldots < w_{n,\tau}\}$ for each embedding $\tau : F \hookrightarrow C$. This data suffices to construct a smooth C-representation $\pi_{sm}(\mathcal{D})$ of $GL_n(F)$ via the classical local Langlands (up to some modifications - see Breuil [5] for further details). For each τ, we let $\pi_{alg,\tau}(\mathcal{D})$ be the irreducible algebraic representation of $GL_n(F)$ of highest weight $(w_{n,\tau} - (n-1), \ldots, w_{1,\tau})$ relative to the upper-triangular Borel. Then let $\pi_{alg}(\mathcal{D}) = \otimes_{\tau \in Hom(F,C)} \pi_{alg,\tau}(\mathcal{D})$, with $GL_n(F)$ acting diagonally. We can then form $BS(\mathcal{D}) := \pi_{sm}(\mathcal{D}) \otimes_C \pi_{alg}(\mathcal{D})$.

Also, any p-adic potentially semistable representation ρ of $Gal(\overline{\mathbb{Q}}_p/F)$ on an n-dimensional C-vector space, V, gives rise to a Weil-Deligne representation $WD(\rho)$ and tuples of integers $HT_\tau(\rho)$ for each embedding $\tau : F \hookrightarrow C$, as follows. Let F' be a finite Galois extension of F such that $V\mid_{Gal(\overline{\mathbb{Q}}_p/F')}$ is semistable. Set

$$D := (B_{st} \otimes_{\mathbb{Q}_p} V)^{Gal(\overline{\mathbb{Q}}_p/F')} \otimes_{F_0' \otimes C} C$$

where B_{st} is Fontaine's semistable period ring, F_0' is the maximal unramified subfield in F' and $F_0' \hookrightarrow C$ is any embedding.

It is an n-dimensional C-vector space endowed with a nilpotent endomorphism N coming from the one on B_{st}. We define $r(w)$ on D by $r(w) := \varphi^{-d(w)} \circ \overline{w}$ where w is any element in the Weil group of F, $\overline{w}$ is its image in $Gal(F'/F)$, $d(w) \in f\mathbb{Z}$ is the unique integer such that the image of w in $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is the $d(w)$-th power of the absolute arithmetic Frobenius, and φ is the semilinear endomorphism coming from the action of Frobenius on B_{st} (as $\varphi^{-d(w)} \circ \overline{w}$ is $F_0' \otimes C$-linear, $r(w)$ goes down to D). This gives $WD(\rho)$. $HT_\tau(\rho)$ are just its various Hodge-Tate weights.

Conjecture 1.1. *(Breuil and Schneider [6]) There exists a p-adic n-dimensional potentially semi-stable representation ρ of $Gal(\overline{\mathbb{Q}}_p/F)$ such that*

$$\mathcal{D} = (WD(\rho), HT_\tau(\rho))_\tau$$

if and only if $BS(\mathcal{D})$ admits a $GL_n(F)$-invariant norm.

The "if" part is completely known, and is due to Y. Hu (Hu [15]). The "only if" part remains open.

Note that the existence of a G-equivariant norm is equivalent to the existence of a separated lattice: Given a norm $||\cdot||$, the unit ball is a lattice. Conversely, given a lattice Λ, its gauge $||x|| = q_C^{-v_\Lambda(x)}$, where $v_\Lambda(x) = \sup\{v \mid x \in \pi_C^v \Lambda\}$. Thus we are looking for integral structures in $BS(\rho)$.

1.3. Other reductive groups and the case of $G = U_3(F)$

While many attempts were made in order to find criteria for the existence of integral structures in representations of $GL_2(F)$, where F is a finite extension of $\mathbb{Q}_p$, and towards the proof of the Breuil-Schneider conjecture, which concerns the case of $GL_n(F)$, very little is known about the correspondence for other reductive groups.

In fact, only in (Große-Klönne [14]) arbitrary split reductive groups were treated for the first time. However, we are not familiar with any work done concerning non-split reductive groups, and the unitary group in particular.

In such cases, it is still possible to consider the locally algebraic representations of $G = U_3(F)$ over a p-adic field C, and ask whether they admit a G-invariant norm.

One of the reasons for choosing $U_3(F)$ over $U_2(F)$, for example, is the possibility to learn from it new insights that will help us better understand the case of $GL_3(F)$. Even though the case of $U_2(F) = U(1,1)(F)$ seems simpler, and maybe close to the case of $SL_2(F)$, we have not explored this case, and are not aware of efforts made in this direction.

In the case of $GL_n(F)$, Hu (in Hu [15]) shows that the requirement, in the Breuil-Schneider conjecture, that there exists such a p-adic n-dimensional potentially semi-stable Galois representation ρ, is equivalent to a condition formulated by Emerton (in Emerton et al. [11]), stated purely in terms of the reductive group.

Explicitly, let G be a reductive group, and let P be a parabolic subgroup of G with unipotent radical N and Levi component M. Let N_0 be some compact open subgroup of N, and let Z_M be the center of M. Write $Z_M^+ := \{z \in Z_M \mid zN_0z^{-1} \subset N_0\}$. Let δ denote the modulus character of P, which is trivial on N, hence induces a character on $M = P/N$, denoted also by δ. Concretely, $\delta(m) = [mN_0m^{-1} : N_0]$. Let $J_P(V)$ denote Emerton's Jacquet module (with respect to P) of a representation V. If $V = \pi_{alg} \otimes \pi_{sm}$, we have

$$J_P(V) = \pi_{alg}^N \otimes_C (res_P^G \pi_{sm})_N \delta^{1/2}$$

We then have the following Lemma in Emerton et al. [11].

Lemma 1.2. *Let χ be a locally algebraic C-valued character of Z_M. If the χ-eigenspace of $J_P(V)$ is nonzero, and V admits a G-invariant norm, then*

$$\left|\chi(z)\delta^{-1}(z)\right| \leq 1 \quad \forall z \in Z_M^+$$

This criterion is equivalent, in the case of $GL_n(F)$, to the existence of ρ. As this criterion is formulated purely in terms of the reductive group G, it gives rise to a generalization of the Breuil-Schneider conjecture to arbitrary reductive groups.

Conjecture 1.3. *Assume that for any locally algebraic character $\chi : Z_M \to C^\times$ such that $J_P(V)_\chi \neq 0$, one has*

$$\left|\chi(z)\delta^{-1}(z)\right| \leq 1 \quad \forall z \in Z_M^+$$

and that the central character of V is unitary. Then V admits a G-invariant norm.

1.4. Progress on the Breuil Schneider conjecture

- Note that the central character of $BS(\rho)$ always attains values in $\mathcal{O}_C^\times$. Sorensen (Sorensen [25]) has proved for any connected reductive group G defined over $\mathbb{Q}_p$, that if π_{alg} is an irreducible algebraic representation of $G(\mathbb{Q}_p)$, and π_{sm} is an essentially discrete series representation of $G(\mathbb{Q}_p)$, both defined over C, then $\pi_{alg} \otimes_C \pi_{sm}$ admits a $G(\mathbb{Q}_p)$-invariant norm if and only if its central character is unitary.

- Recently there has been spectacular progress on the BS conjecture for $GL_n(F)$ in the principal series case, which is the deepest, by joint work of Caraiani, Emerton, Gee, Geraghty, Paskunas and Shin (Caraiani et al. [8]). Using global methods, they construct a candidate Π for a p-adic local Langlands correspondence for $GL_n(F)$ and are able to say enough about it to prove new cases of the conjecture. Their conclusion is even somewhat stronger than the existence of a norm on $BS(\rho)$, in that it asserts admissibility.

Both works employ the usage of global methods, and as this is a question of local nature, one hopes to find some local method to recover these results. There has also been some progress employing local methods, which yields results also for finite extensions of $\mathbb{Q}_p$, namely:

- For $GL_2(F)$, Vigneras (Vignéras [29]) constructed an integral structure in tamely ramified smooth principal series representations, satisfying the assumption that they arise from p-adic potentially semistable Galois representations.

- For $GL_2(F)$, following the methods of Breuil over $\mathbb{Q}_p$, de Ieso (De Ieso [9]) used compact induction together with the action of the spherical Hecke algebra to produce a separated lattice in $BS(\rho)$ where $BS(\rho)$ is an unramified locally algebraic principal series representation, under some technical p-smallness condition on the weight.

- For $GL_2(F)$, in a joint work with Kazhdan and de Shalit (Assaf et al. [3]), we have used p-adic Fourier theory for the Kirillov model to get integral structures if $BS(\rho)$ is a tamely ramified smooth principal series or an unramified locally algebraic principal series, satisfying the assumption that they arise from p-adic potentially semistable Galois representations.

- It is possible to generalize the assertions of the Breuil-Schneider conjecture to arbitrary split reductive groups. The conjecture then asserts the existence of a G-invariant norm when certain conditions, which can be expressed purely in terms of the reductive group, G, are met. For general split reductive groups, Große-Klonne (Große-Klönne [14]) looked at the universal module for the spherical Hecke algebra, and was able to show some instances of this generalization for unramified principal series, again under some p-smallness condition on the Coxeter number (when $F = \mathbb{Q}_p$) plus other technical assumptions.

We will employ all of the above methods in our present paper.

1.5. Our work

Let p be a prime number. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, a finite extension F of $\mathbb{Q}_p$ and a quadratic extension E of F, both in $\overline{\mathbb{Q}}_p$. We choose a finite extension C of $\mathbb{Q}_p$, which will serve as the field of coefficients for our representations. We assume that C contains the normal closure of $E/\mathbb{Q}_p$, so that

$$|Hom(E, C)| = [E : \mathbb{Q}_p].$$

Let V be a 3 dimensional vector space over E. Let $\sigma \in Gal(E/F)$ be the nontrivial involution. We shall often denote $\overline{x} = \sigma(x)$ for $x \in E$.
We shall denote by E^1 the norm one elements in E, i.e.

$$E^1 = U_1(F) = \{x \in E \mid x\overline{x} = 1\}.$$

Denote by θ the Hermitian form on V represented by the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ with respect to the standard basis, which we will denote by l_{-1}, l_0, l_1. Explicitly

$$(u, v) = {}^t\overline{v}\theta u$$

Let $G = U_3(F) = U(\theta) = \{g \in GL_3(E) \mid {}^t\overline{g}\theta g = \theta\}$ be the unitary group in three variables over F.
Let B be its Borel subgroup of upper triangular matrices, and let

$$M = \left\{ m_{t,s} := \begin{pmatrix} t & & \\ & s & \\ & & \overline{t}^{-1} \end{pmatrix} \mid t \in E^\times, s \in E^1 \right\} \simeq E^\times \times E^1$$

be the maximal torus of diagonal matrices contained in it. Let $\chi : E^\times \to C^\times$ and $\chi_1 : E^1 \to C^\times$ be smooth characters, and define $\chi_M(m_{t,s}) = \chi(t)\chi_1(s)$. We denote by $\chi_B : B \to C^\times$ its inflation to B, and write also $\chi_B = \chi \otimes \chi_1$. Denote

$$Ind_B^G(\chi_B) = \left\{ f : G \to C \mid \begin{array}{c} \exists U_f \text{ open s.t. } f(bgk) = \chi_B(b)f(g) \\ \forall g \in G, \quad b \in B, \quad k \in U_f \end{array} \right\}$$

with the group G acting by right translations, namely $(gf)(x) = f(xg)$ for all $x, g \in G$ and $f \in Ind_B^G(\chi_B)$.

For any $\tau \in Hom_{alg}(E,C)$, let d_τ, $0 \le a_\tau, b_\tau$ be integers. Denote $\underline{a} = (a_\tau), \underline{b} = (b_\tau), \underline{d} = (d_\tau) \in \mathbb{Z}^{Hom_{alg}(E,C)}$, and let $\rho_{\underline{a},\underline{b},\underline{d}}$ be the irreducible algebraic representation associated to them by subsection 3.1.

Let $\pi = Ind_B^G(\chi \otimes \chi_1) \otimes \rho_{\underline{a},\underline{b},\underline{d}}$. We shall show that any locally algebraic principal series C-representation is of this form, hence we are interested in the existence of G-invariant norms on these representations.

We restrict ourselves to two cases: either χ, χ_1 are unramified and the algebraic weights $\underline{a}, \underline{b}$ are small, or χ, χ_1 are tamely ramified and there is no algebraic part (the representation is smooth), and give a necessary and sufficient criterion in these cases for the existence of a G-invariant norm.

For the smooth case ($\underline{a} = \underline{b} = 0$), we shall employ the method of coefficient systems on the Bruhat-Tits tree of U_3, introduced by Vigneras in (Vignéras [29]), while for the unramified locally algebraic case, we will employ the methods introduced in (De Ieso [9]) and in (Große-Klönne [14]).

In section 2, we recall some basic properties of the group $G = U_3(F)$ and we review briefly the construction and properties of the Bruhat-Tits tree associated to it.

In section 3, we classify locally algebraic representations of G, and introduce stable $\mathcal{O}_C G$-modules which we conjecture to be integral structures in such representations. Here we show the connection between compact induction and the principal series representations in terms of the spherical Hecke algebra.

In Section 4, we focus on unramified locally algebraic principal series representations, and prove the first of our main theorems - a necessary and sufficient criterion for such a representation to admit a G-invariant norm.

In Section 5, we introduce the concept of G-equivariant coefficient systems on the tree, and show that they are equivalent to "diagrams" - fundamental systems in the tree which suffice to describe the entire coefficient system (see Definition 5.8) . We further show the intimate connection between induced representations of G, and G-equivariant coefficient systems on the tree.

In Section 6, we give a local criterion for integrality of the 0-th homology of certain coefficient systems, as a representation of G. We further refine the criterion, and use the result of Schneider and Stuhler (Schneider and Stuhler [21]), to show that any irreducible locally algebraic representation can be attained as the 0-th homology of some coefficient system on the tree.

In Section 7, we focus our attention on the case of smooth tamely ramified principal series representations, and prove the second of our main theorems - a necessary and sufficient criterion for such a representation to admit a G-invariant norm.

In Section 8, we briefly discuss the relevance of these results to representations over a finite field of characteristic p (mod p representations).

1.6. Notations

Let $p > 2$ be an odd prime number. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, a finite extension, F, of $\mathbb{Q}_p$, contained in $\overline{\mathbb{Q}}_p$, and a quadratic extension, E, of F, contained in $\overline{\mathbb{Q}}_p$. We choose a finite extension, C, of $\mathbb{Q}_p$, which will serve as the field of coefficients for our representations. We choose it such that it satisfies the following condition:

$$|Hom_{alg}(E, C)| = [E : \mathbb{Q}_p]$$

For any field K (which could be either F, E or C), we denote by $\mathcal{O}_K$ its ring of integers, by $\mathfrak{p}_K$ its maximal ideal, by π_K a uniformizer of $\mathcal{O}_K$, by k_K its residue field, and by q_K its cardinality.

For any field K admitting an automorphism σ of order 2 (here K will be either E or k_E), denote $K^- = \{x \in K \mid \sigma(x) = -x\}$ and $K^1 = \{x \in K \mid x \cdot \sigma(x) = 1\}$.

We let $q = q_E = p^f$, and denote by e the ramification index of E over $\mathbb{Q}_p$, so that $ef = [E : \mathbb{Q}_p]$. The p-adic valuation val_E on $\overline{\mathbb{Q}}_p$ is normalized by $val_E(\pi_E) = 1$, and we set $|x| = q_E^{-val_E(x)}$ for $x \in \overline{\mathbb{Q}}_p$. We also let $\pi = \pi_E$. Thus $|\pi| = q^{-1}$.

Let $V_0 = \mathcal{O}_E^3$, and let l_{-1}, l_0, l_1 be the standard basis. Let $V = V_0 \otimes_{\mathcal{O}_E} E$ be its scalar extension to a vector space over E.

Let $\sigma \in Gal(E/F)$ be the nontrivial involution. We shall often write $\overline{x} = \sigma(x)$ for $x \in E$.

Denote by θ the Hermitian form on V_0 repersented by the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ with respect to l_{-1}, l_0, l_1. Explicitly

$$(u, v) = {}^t\overline{v}\theta u, \quad \theta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Let $\mathbf{G} = \mathbf{U}_3 = \mathbf{U}(\theta)$ be the unitary group scheme over $\mathcal{O}_F$. Thus, for any $\mathcal{O}_F$-algebra A

$$\mathbf{G}(A) = \mathbf{U}_3(A) = \{g \in GL_3(\mathcal{O}_E \otimes_{\mathcal{O}_F} A) \mid {}^t\overline{g}\theta g = \theta\}$$

is all the invertible linear transformations on $V_0 \otimes_{\mathcal{O}_F} A$ which preserve θ.

Denote by $\mathbf{B} = \mathbf{MN}$ the Borel subgroup of upper triangular matrices in $\mathbf{G}$, where $\mathbf{M}$ is the maximal torus of diagonal matrices contained in it, and $\mathbf{N}$ is its unipotent radical.

Denote $G = \mathbf{G}(F)$, let $K_0 = GL_3(\mathcal{O}_E) \cap G$, and let $B = \mathbf{B}(F)$, $M = \mathbf{M}(F)$, $N = \mathbf{N}(F)$. Then

$$M = \left\{ m_{t,s} := \begin{pmatrix} t & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \overline{t}^{-1} \end{pmatrix} \mid t \in E^\times, s \in E^1 \right\} \simeq E^\times \times E^1$$

Let $\chi : E^\times \to C^\times$ and $\chi_1 : E^1 \to C^\times$ be smooth characters, and define $\chi_M(m_{t,s}) = \chi(t)\chi_1(s)$. We denote by $\chi_B : B \to C^\times$ its inflation to B, and write also $\chi_B = \chi \otimes \chi_1$.

Denote

$$Ind_B^G(\chi_B) = \left\{ f : G \to C \mid \begin{array}{c} \exists U_f \text{ open s.t. } f(bgk) = \chi_B(b) f(g) \\ \forall g \in G, \quad b \in B, \quad k \in U_f \end{array} \right\}$$

with the group G acting by right translations, namely $(gf)(x) = f(xg)$ for all $x, g \in G$ and $f \in Ind_B^G(\chi_B)$.

We also denote

$$s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \pi^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{\pi} \end{pmatrix}, \quad \beta = \alpha s = \begin{pmatrix} 0 & 0 & \pi^{-1} \\ 0 & 1 & 0 \\ \overline{\pi} & 0 & 0 \end{pmatrix}$$

Remark 1.4. Although we have $s = \theta$, the symbol s is used when referring to an element of $GL_3(E)$, whileθ is used when referring to the hermitian form.

For any vector space W over C and any nonnegative integer n, we denote by $T(W)$ its tensor algebra, by $\mathrm{Sym}W = {}^{T(W)}/_{\langle v \otimes w - w \otimes v | v, w \in W \rangle}$ the symmetric algebra over W, and by $\mathrm{Sym}^n W$ the n-th graded piece of $\mathrm{Sym}W$, which is the n-th symmetric power of W. We denote by $W^\star$ the dual vector space.

We shall write $x_1 \cdots x_n$ for the representative of $x_1 \otimes \ldots \otimes x_n \in T(W)$ in $Sym^n W$.

For nonnegative integers a, b, we denote $S(a, b) = \mathrm{Sym}^a C^3 \otimes \mathrm{Sym}^b (C^3)^\star$. Then for any $a, b \geq 1$, there is a natural contraction map $\iota_{a,b} : S(a, b) \to S(a-1, b-1)$, defined by

$$\iota_{a,b}(x_1 \cdots x_a \otimes y_1 \cdots y_b) = \sum_{i=1}^{a} \sum_{j=1}^{b} \langle x_i, y_j \rangle \cdot (x_1 \cdots \hat{x}_i \cdots x_a \otimes y_1 \cdots \hat{y}_j \cdots y_b)$$

We denote by $\rho(a, b) = \ker(\iota_{a,b})$. For completeness, if $ab = 0$, we denote $\rho(a, b) = S(a, b)$ be the entire space.

The space $S(a, b)$, as well as its subspace $\rho(a, b)$, admit natural actions of G.

In fact, for any $d \in \mathbb{Z}$, this action can be twisted by $\det^d$, to obtain a representation which we denote by $\rho(a, b, d)$.

The action will be described explicitly in subsection 2.3. These spaces will be used to describe the irreducible rational representations of G.

We also fix an embedding $\iota : E \hookrightarrow C$, and denote other embeddings by $\tau : E \hookrightarrow C$.

1.7. Main Theorems

Theorem 1.5. *Let $\chi : E^\times \to C^\times$ be an unramified character. For any $\tau \in Hom_{alg}(E, C)$, let d_τ, $0 \leq a_\tau, b_\tau < p$ be integers, let $a = \sum_\tau a_\tau, b = \sum_\tau b_\tau$ and denote*

$$\rho = \bigotimes_{\tau \in Hom_{alg}(E,C)} \rho_\tau, \quad \rho_\tau = \rho(a_\tau, b_\tau, d_\tau) \otimes_{E,\tau} C$$

where $\rho(a_\tau, b_\tau, d_\tau)$ is as above. Then the following are equivalent:

1. $\pi^{a+b}\chi(\pi)^{-1}, \quad q^2\pi^{a+b}\chi(\pi) \in \mathcal{O}_C$ (equivalently $\left|\pi^{-a-b}\right| \leq |\chi(\pi)| \leq |q^{-2}\pi^{-a-b}|$)
2. $Ind_B^G(\chi) \otimes \rho$ admits a G-invariant norm.

Theorem 1.6. *Let $\chi : E^\times \to C^\times$, $\chi_1 : E^1 \to C^\times$ be tamely ramified characters. Then the following are equivalent:*

1. $\chi(\pi)^{-1}, \quad q^2\chi(\pi) \in \mathcal{O}_C$ (equivalently $1 \leq |\chi(\pi)| \leq |q^{-2}|$)
2. $Ind_B^G(\chi \otimes \chi_1)$ admits a G-invariant norm.

Remark 1.7. Simple calculation shows that both theorems are in fact special cases of Conjecture 1.3, specialized to the group $G = U_3(F)$. This follows from the fact that the action of the non-trivial element in the Weyl group of G, w, on a character χ is given by $\chi^w = q^2\chi^{-1}$.

2. The group $U_3(F)$

It has been brought to our attention that some of the material for this section can also be found in Abdellatif [1], which even presents some generalizations for the case of quasi-split group of rank one, and in Koziol and Xu [19].

2.1. The Bruhat-Tits tree of U_3

In (Abramenko and Nebe [2]) it is proved that the Bruhat-Tits tree of $G = U_3(F)$ can be obtained by considering the action of G on the Bruhat-Tits building of $GL_3(E)$.

Let $\mathcal{X}$ be the Bruhat-Tits building of $GL_3(E)$. It is a simplicial complex, which may be describes as follows. (We follow the descriptions given in Abramenko and Nebe [2], Garrett [13]).

Let V be a 3-dimensional vector space over E. Let π be a uniformizer of $\mathcal{O}_E$.

Definition 2.1. Let V be a vector space over E. A *lattice* in V is a finitely generated $\mathcal{O}_E$-module spanning V over E.

We define an equivalence relation on the set of lattices in V as follows.

Definition 2.2. Two lattices L, L' lie in the same *dilation class* if there exists a $\lambda \in E^\times$ such that $L' = \lambda L$.

Then the vertices of $\mathcal{X}$ consist of dilation classes $[L]$ of lattices $L \subset V$. The edges of $\mathcal{X}$ consist of pairs $\{[L_0], [L_1]\}$ where $L_0 \supsetneq L_1 \supsetneq \pi L_0$.

The 2-cells of $\mathcal{X}$ consist of triples $\{[L_0], [L_1], [L_2]\}$ where $L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \pi L_0$.

The group $GL_3(E)$ acts on the left on $\mathcal{X}$, by $g[L] = [gL]$ for any $g \in GL_3(E)$.

Equivalently, we may define

Definition 2.3. A chain of lattices in V, $\ldots \subseteq L_i \subseteq L_{i+1} \subseteq \ldots$ is called *admissible* if the set $\{L_i\}_{i \in \mathbb{Z}}$ is closed under multiplication by integral powers of π.

Then $\mathcal{X}$ is the partially ordered (by inclusion) set of all admissible chains of lattices. (see Abramenko and Nebe [2], 3.3).

Let $V = E^3$, and let $\{l_{-1}, l_0, l_1\}$ be the standard basis. Let $(\cdot,\cdot)$ be the Hermitian form on V defined by

$$(u,v) = {}^{t}\overline{v} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot u \quad \forall u,v \in V$$

Let $L_0 = \mathcal{O}_E l_{-1} + \mathcal{O}_E l_0 + \mathcal{O}_E l_1$. Given a lattice $L = gL_0$, we see that $\det(\pi^k g) = \pi^{3k}\det(g)$, hence $val_E(\det(\pi^k g)) = 3k + val_E(\det g)$. It follows that for any class $[L]$, there exists a lattice $L = gL_0$ with $val_E(\det g) \in \{0,1,2\}$. Thus we may associate a number (or type) to each vertex.

Definition 2.4. Let $v = [L]$ be a vertex of $\mathcal{X}$. Let $g \in GL_3(E)$ such that $L = gL_0$, then the *type* of v is $val_E(\det g) \mod 3$.

Lemma 2.5. *The hermitian form $(\cdot,\cdot)$ induces a non-type-preserving automorphism σ of $\mathcal{X}$, of order 2, by setting*

$$L^\sharp = \{v \in V \mid (v,l) \in \mathcal{O}_E \quad \forall l \in L\}, \quad \sigma([L]) = [L^\sharp]$$

Proof. If $[L'] = [L]$, there exists some $\lambda \in E^\times$ such that $L' = \lambda L$, hence

$$(L')^\sharp = \{v \in V \mid (v,l) \in \mathcal{O}_E \quad \forall l \in L'\} = \{v \in V \mid (v,\lambda l) \in \mathcal{O}_E \quad \forall l \in L\} =$$

$$= \{v \in V \mid (\overline{\lambda} v, l) \in \mathcal{O}_E \quad \forall l \in L\} = \overline{\lambda}^{-1} \cdot L^\sharp$$

showing that $\left[(L')^\sharp\right] = [L^\sharp]$, hence σ is well defined. Furthermore, we have

$$\left(L^\sharp\right)^\sharp = \{v \in V \mid (v,l') \in \mathcal{O}_E \quad \forall l' \in L^\sharp\} = L$$

showing that $\sigma^2 = 1$. □

Remark 2.6. Note that if $L = gL_0$ for some $g \in GL_3(E)$, and $L^\sharp = g^\sharp L_0$ for some $g^\sharp \in GL_3(E)$, we have

$${}^{t}\overline{g^\sharp} \cdot g \in GL_3(\mathcal{O}_E)$$

hence

$$\det(g^\sharp)\cdot\det(g) \in \mathcal{O}_E^\times \Rightarrow val_E(\det(g^\sharp)) = -val_E(\det(g))$$

showing that vertices of type 0 are fixed by σ, while vertices of types $1,2$ are paired by σ.

This allows us to assign types also for orbits of σ - fixed points will be of type 0, and orbits of length two will be of type 1.

This involution induces also a map on chains of lattices, defined by

$$\mathcal{L} = \ldots \subseteq L_i \subseteq L_{i+1} \subseteq \ldots, \quad \mathcal{L}^\sharp = \ldots \subseteq L_{i+1}^\sharp \subseteq L_i^\sharp \subseteq \ldots$$

Definition 2.7. An admissible chain of lattices $\mathcal{L}$ is called #-admissible if $\mathcal{L}^{\#} = \mathcal{L}$.

This suggests the following result.

Proposition 2.8. *(Abramenko and Nebe [2], §8) The Bruhat-Tits tree, $\mathcal{T}$, of $U_3(F)$ is the partially ordered set (by inclusion) of #-admissible chains of lattices.*

For example, in the *standard chamber* of $\mathcal{X}$, consisting of

$$v_0 = [L_0], v_1 = [L_1] = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix} L_0 \right], v_2 = [L_2] = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix} L_0 \right]$$

the admissible chain of lattices $\ldots \subseteq \pi L_0 \subseteq L_0 \subseteq \pi^{-1} L_0 \subseteq \ldots$ representing v_0 is also #-admissible, since $(\pi^k L_0)^\sharp = \pi^{-k} L_0$.

However, the admissible chains of lattices $\ldots \subseteq \pi L_1 \subseteq L_1 \subseteq \pi^{-1} L_1 \subseteq \ldots$, and $\ldots \subseteq \pi L_2 \subseteq L_2 \subseteq \pi^{-1} L_2 \subseteq \ldots$ representing v_1, v_2 are not #-admissible since $L_1^\sharp = \pi^{-1} L_2$.

Instead, one sees that the admissible chain of lattices $\ldots \subseteq \pi L_1 \subseteq L_2 \subseteq L_1 \subseteq \pi^{-1} L_2 \subseteq \pi^{-1} L_1 \subseteq \ldots$ representing the edge (v_1, v_2) is #-admissible.

This shows that the edge (v_1, v_2) in $\mathcal{X}$ will contract to a single vertex in $\mathcal{T}$.

More generally, there are two types of minimal #-admissible chains of lattices, which correspond to the vertices of the tree $\mathcal{T}$ -

Either $L_{i+1} = \pi L_i$ and $L_i^\sharp = L_i$, in which case this is just an original vertex of $\mathcal{X}$, or there exists i such that $L_{i+1}^\sharp = L_i$, in which case it is the contraction of an edge of $\mathcal{X}$.

The edges of $\mathcal{T}$ are the contracted simplices of $\mathcal{X}$ - triangles contracted along one edge.

Let $\mathcal{T}_k$ denote the k-simplices of $\mathcal{T}$, so that $\mathcal{T}_0$ are the vertices, and $\mathcal{T}_1$ are the edges. Let $\widehat{\mathcal{T}_1}$ denote the oriented edges.

Definition 2.9. If L is a lattice satisfying $L \subseteq L^{\#} \subseteq \pi^{-1} L$, we say that L is a *standard lattice.*

Remark 2.10. We may identify the vertices of the tree $\mathcal{T}$ with the standard lattices, where each represents an equivalence classes (under # and homothety) of lattices.

We then have two types of vertices - the vertices represented by standard lattices with $L^\sharp = L$, and the vertices represented by standard lattices $L^\sharp \supsetneq L \supsetneq \pi L^\sharp$.

Let

$$\mathcal{T}_0^0 = \{v \in \mathcal{T}_0 \mid v = [L], \quad L = L^\sharp\}, \quad \mathcal{T}_0^1 = \{v \in \mathcal{T}_0 \mid v = ([L], [L^\sharp]), \quad L \neq L^\sharp\}$$

We call $\mathcal{T}_0^0$ vertices of type 0, and $\mathcal{T}_0^1$ vertices of type 1.

Two such vertices, $L_0 = L_0^\sharp$, and $L_1^\sharp \supseteq L_1 \supseteq \pi L_1^\sharp$ are connected with an edge if $L_1^\sharp \supseteq L_0 \supseteq L_1$.

We have also the following description of the apartments in $\mathcal{T}$.

Proposition 2.11. *(Abramenko and Nebe [2], §6) The apartments of $\mathcal{T}$ correspond to hyperbolic frames, i.e. pairs of lines $Ev_1, Ev_2 \subset V$ such that $(v_1, v_2) \neq 0, (v_1, v_1) = (v_2, v_2) = 0$. The corresponding apartment consists of all classes of lattices that admit a basis contained in $\{v_1, v_2\} \cup \langle v_1, v_2 \rangle^\perp$.*

Definition 2.12. We set $v_0 = [L_0]$ where $L_0 = \mathcal{O}_E l_{-1} + \mathcal{O}_E l_0 + \mathcal{O}_E l_1$, and $v_1 = [L_1]$ where

$$L_1 = \mathcal{O}_E \cdot l_{-1} + \mathcal{O}_E \cdot l_0 + \pi \mathcal{O}_E \cdot l_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix} \cdot L_0$$

Then

$$L_1^\sharp = \pi^{-1} \mathcal{O}_E \cdot l_{-1} + \mathcal{O}_E \cdot l_0 + \mathcal{O}_E \cdot l_1 \supset L_0 \supset L_1 \supset \pi L_1^\sharp$$

so that L_0, L_1 are standard lattices, with $v_0 \in \mathcal{T}_0^0, v_1 \in \mathcal{T}_0^1$, and $e_{01} = (v_0, v_1) \in \mathcal{T}_1$ is an edge.

Then e_{01} will be called the *standard chamber* in the apartment corresponding to the pair El_{-1}, El_1, which we will refer to as the *standard apartment*.

By (Abramenko and Nebe [2], Lemma 17), we see that G acts transitively on the set of chambers in $\mathcal{T}$. Further, for any $g \in G$, we see that

$${}^t\overline{g}\theta g = \theta \Rightarrow Nm_{E/F}(\det g) = \det(g) \cdot \det(\overline{g}) = 1, \quad \theta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

hence $val_E(\det g) = 0$, showing that G preserves types.

It then follows that

Corollary 2.13. *For any edge $(u_0, u_1) \in \mathcal{T}$, with $u_0 \in \mathcal{T}_0^0$ of type 0 and $u_1 \in \mathcal{T}_0^1$ of type 1, there exists $g \in G$ such that $u_0 = gv_0$ and $u_1 = gv_1$.*

2.2. Structure of U_3

We denote by K_0 the stabilizer in G of v_0, and by K_1 the stabilizer in G of v_1; the intersection $I = K_0 \cap K_1$ is the stabilizer in G of $e_{01} = (v_0, v_1)$.

By construction (see Tits [27]), the groups K_0, K_1 are representatives of the two conjugacy classes of maximal compact open subgroups of G.

Since G preserves types, the stabilizer of an edge is the same as the stabilizer of an oriented edge.

Since the action of G on $\mathcal{T}$ is transitive on each type of vertices, we note that the vertices of type $i \in \{0, 1\}$ are in bijection with left cosets G/K_i.

Proposition 2.14. *Representing the elements of G in the basis l_{-1}, l_0, l_1, we have*

$$K_0 = GL_3(\mathcal{O}_E) \cap G, \quad K_1 = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \pi^{-1}\mathcal{O}_E \\ \mathfrak{p}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathcal{O}_E \end{pmatrix} \cap G$$

$$K_0 \cap K_1 = I = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathfrak{p}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathcal{O}_E \end{pmatrix} \cap G$$

Proof. Let $g \in K_0$. Recall that $L_0 = \mathcal{O}_E l_{-1} + \mathcal{O}_E l_0 + \mathcal{O}_E l_1$. Then, as $[gL_0] = [L_0]$, and $\det(g) \in \mathcal{O}_E^\times$, we must have $gL_0 = L_0$, whence $gl_{-1}, gl_0, gl_1 \in L_0 = \mathcal{O}_E l_{-1} + \mathcal{O}_E l_0 + \mathcal{O}_E l_1$.

It follows that $g \in M_3(\mathcal{O}_E)$. Since $g \in G \subset GL_3(E)$ and $g^{-1}L_0 = L_0$, by symmetry we see that $g \in GL_3(\mathcal{O}_E) \cap G$. Conversely, if $g \in GL_3(\mathcal{O}_E) \cap G$, then clearly $gl_{-1}, gl_0, gl_1 \in L_0$, showing that $gL_0 \subset L_0$, and as $g \in GL_3(\mathcal{O}_E)$, we see that $g^{-1} \in GL_3(\mathcal{O}_E) \cap G$, hence $g^{-1}L_0 \subset L_0$, so $L_0 \subset gL_0$, showing equality. Therefore $K_0 = GL_3(\mathcal{O}_E) \cap G$.

Next, let $g \in K_1$. Since $[gL_1] = [L_1]$ and $\det(g) \in \mathcal{O}_E^\times$, we must have $gL_1 = L_1$ and furthermore $gL_1^\sharp = L_1^\sharp$.

As $L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix} \cdot L_0$, we see that $L_1^\sharp = \begin{pmatrix} \pi^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot L_0$, and, with the obvious definitions of g_{-1}, g_1,

$$g_1 L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi^{-1} \end{pmatrix} g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix} L_0 = L_0$$

$$g_{-1} L_0 = \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g \begin{pmatrix} \pi^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} L_0 = L_0$$

showing that $g_1, g_{-1} \in K_0$, whence

$$g \in \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix} K_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi^{-1} \end{pmatrix} \cap \begin{pmatrix} \pi^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} K_0 \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \mathcal{O}_E & O_E & \pi^{-1}\mathcal{O}_E \\ \mathcal{O}_E & \mathcal{O}_E & \pi^{-1}\mathcal{O}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathcal{O}_E \end{pmatrix} \cap \begin{pmatrix} \mathcal{O}_E & \pi^{-1}O_E & \pi^{-1}\mathcal{O}_E \\ \mathfrak{p}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathfrak{p}_E & \mathcal{O}_E & \mathcal{O}_E \end{pmatrix} \cap G =$$

$$= \begin{pmatrix} \mathcal{O}_E & O_E & \pi^{-1}\mathcal{O}_E \\ \mathfrak{p}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathcal{O}_E \end{pmatrix} \cap G$$

Conversely, if $g \in \begin{pmatrix} \mathcal{O}_E & O_E & \pi^{-1}\mathcal{O}_E \\ \mathfrak{p}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathcal{O}_E \end{pmatrix} \cap G$, then g_1, g_{-1}, defined as above, satisfy $g_1, g_{-1} \in K_0$,
hence $g_1 L_0 = L_0, g_{-1} L_0 = L_0$, showing that $gL_1 = L_1$ and $gL_1^\sharp = L_1^\sharp$. □

The following proposition is a special case of (Tits [27], 3.3.3). It follows from the fact that K_i acts transitively on the vertices at distance $2n$ from v_i.

Proposition 2.15. *(Cartan decomposition) If we denote* $\alpha = \begin{pmatrix} \pi^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{\pi} \end{pmatrix}$, *and let* $i \in \{0, 1\}$ *then*

$$G = \coprod_{n \in \mathbb{Z}_{\geq 0}} K_i \alpha^{-n} K_i.$$

The following proposition is a special case of (Tits [27], 3.3.1), but since our case is much simpler, we prove it directly.

Proposition 2.16. *(Iwahori decomposition) We have*

$$K_0 = I \coprod IsI, \quad K_1 = I \coprod I\beta I$$

where

$$s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \beta = \alpha s = \begin{pmatrix} 0 & 0 & \pi^{-1} \\ 0 & 1 & 0 \\ \overline{\pi} & 0 & 0 \end{pmatrix}.$$

Proof. If E/F is unramified, consider the natural reduction map $\rho_0 : K_0 = GL_3(\mathcal{O}_E) \cap G \to \mathbf{G}(k_F)$ defined by reducing the entries modulo π. Then $I = \rho_0^{-1}(\mathbf{B}(k_F))$ is the preimage of the parabolic subgroup of upper triangular matrices over the residue field, and considering the Bruhat decomposition over the residue field, we see that

$$\mathbf{G}(k_F) = \mathbf{B}(k_F) \coprod \mathbf{B}(k_F) s \mathbf{B}(k_F)$$

Taking preimages under ρ_0, we see that

$$K_0 = GL_3(\mathcal{O}_E) \cap G = \rho_0^{-1}(\mathbf{G}(k_F)) = I \coprod IsI$$

For K_1, we consider the group

$$\mathbf{H}(k_F) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{pmatrix} \mid c \in k_E^1, \quad a\overline{d}, b\overline{e} \in k_E^-, \quad \overline{a}e + \overline{d}b = 1 \right\} \leq \mathbf{G}(k_F)$$

where $k_E^- = \{x \in k_E \mid x + \overline{x} = 0\}$ and $k_E^1 = \{x \in k_E \mid x \cdot \overline{x} = 1\}$.

Consider the natural reduction map $\rho_1 : K_1 \to \mathbf{H}(k_F)$ defined by

$$\rho_1 \begin{pmatrix} a & b & \pi^{-1}c \\ \pi d & e & f \\ \pi g & \pi h & i \end{pmatrix} = \begin{pmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & i \end{pmatrix} \mod \pi$$

Then $I = \rho_1^{-1}(\mathbf{B}_H(k_F))$ where $\mathbf{B}_H(k_F) \subset \mathbf{H}(k_F)$ is the parabolic subgroup of lower triangular matrices in H. Considering the Bruhat decomposition of $H(k_F)$ we see that

$$\mathbf{H}(k_F) = \mathbf{B}_H(k_F) \coprod \mathbf{B}_H(k_F) s \mathbf{B}_H(k_F)$$

Taking preimages under ρ_1, we see that

$$K_1 = \rho_1^{-1}(\mathbf{H}(k_F)) = I \coprod I \rho_1^{-1}(s) I = I \coprod I \beta I$$

If E/F is ramified, we consider similarly the natural reductions $\rho_0 : K_0 \to \mathbf{O_3}(k_F)$ and $\rho_1 : K_1 \to \mathbf{H}'(k_F)$, where

$$\mathbf{O_3}(k_F) = \{g \in GL_3(k_F) \mid {}^{\mathrm{t}}g \theta g = \theta\}$$

and

$$\mathbf{H}'(k_F) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & \pm 1 & 0 \\ c & 0 & d \end{pmatrix} \mid ad - bc = 1 \right\} \leq \mathbf{O_3}(k_F)$$

Letting $\mathbf{B}'(k_F) \leq \mathbf{O_3}(k_F)$ and $\mathbf{B}'_H(k_F) \leq \mathbf{H}'(k_F)$ be the parabolic subgroups of upper and lower triangular matrices, respectively, over the residue field, we may consider the Bruhat decompositions of $\mathbf{O_3}(k_F)$ and $H'(k_F)$ to obtain

$$\mathbf{O_3}(k_F) = \mathbf{B}'(k_F) \coprod \mathbf{B}'(k_F) s \mathbf{B}'(k_F), \quad \mathbf{H}'(k_F) = \mathbf{B}'_H(k_F) \coprod \mathbf{B}'_H(k_F) s \mathbf{B}'_H(k_F)$$

Again, $I = \rho_0^{-1}(\mathbf{B}'(k_F)) = \rho_1^{-1}(\mathbf{B}'_H(k_F))$ so taking preimages we obtain the required result. $\square$

The geometric meaning is that the stabilizer of an edge $e = (o(e), t(e)) \in \mathcal{T}_1$ acts transitively on the remaining edges starting at $o(e)$.

Let $\mathbf{N}$ be the unipotent radical of $\mathbf{G}$. Then $N = \mathbf{N}(F)$ is given by

$$N = \left\{ n_{b,z} := \begin{pmatrix} 1 & b & z \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{pmatrix} \mid b, z \in E, \quad z + \bar{z} + b\bar{b} = 0 \right\}$$

The following is a consequence of (Tits [27] 1.15 and 3.5, see also Bruhat and Tits [7] 4.4), but as in this specific case, it is much simpler, we provide a straightforward proof.

Lemma 2.17. *(Iwasawa decomposition) One has $G = BK_0 = BK_1$.*

Proof. First note, by the Bruhat decomposition, that $G = B \coprod BsN$, hence it is enough to show that for any $n \in N$, $sn \in BK_0 \cap BK_1$.

Let $n = n_{b,z} \in N$. If $z \in \mathcal{O}_E$, then $b\overline{b} = -z - \overline{z} \in \mathcal{O}_E \cap F = \mathcal{O}_F$, hence $b \in \mathcal{O}_E$. In that case, note that

$$sn = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -\overline{b} \\ 1 & b & z \end{pmatrix} = \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{\pi}^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \pi^{-1} \\ 0 & 1 & -\overline{b} \\ \overline{\pi} & \overline{\pi}b & \overline{\pi}z \end{pmatrix} \in BK_1 \cap K_0$$

Else, we have $z^{-1} \in \mathfrak{p}_E$ and as $b\overline{b} = -z - \overline{z}$, also $z^{-1}b, z^{-1}\overline{b} \in \mathfrak{p}_E$. In this case, we note that

$$sn = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -\overline{b} \\ 1 & b & z \end{pmatrix} = \begin{pmatrix} \overline{z}^{-1} & \overline{z}^{-1}b & 1 \\ 0 & 1 & -\overline{b} \\ 0 & 0 & z \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ z^{-1}\overline{b} & -z^{-1}\overline{z} & 0 \\ z^{-1} & z^{-1}b & 1 \end{pmatrix} \in BK_0 \cap BK_1$$

□

For later use, we shall need a decomposition of G to left cosets of K_0. For that we introduce some notation.

Let $N_0 = \mathbf{N}(\mathcal{O}_F) = \{n_{b,z} \in N \mid b, z \in \mathcal{O}_E\}$, and for any $r \in \mathbb{N}$,

$$N_{2r} = \{n_{b,z} \in N \mid b \in \pi^r \mathcal{O}_E, \quad z \in \pi^{2r}\mathcal{O}_E\}$$

and

$$N_{2r-1} = \{n_{b,z} \in N \mid b \in \pi^r \mathcal{O}_E, z \in \pi^{2r-1}\mathcal{O}_E\}.$$

Further, for any $r \in \mathbb{N}$, denote $\overline{N}_r = sN_r s$ for the filtration on the opposite unipotent radical. We then have

Proposition 2.18. *For any $n \geq 0$, let R_n be a system of representatives for N_0/N_{2n}, and for $n \geq 1$ let $\overline{R}_n$ be a system of representatives for $\overline{N}_1/\overline{N}_{2n}$. Then*

$$G = \left(\coprod_{\eta \in R_n, n \geq 0} \eta\alpha^{-n}K_0 \right) \coprod \left(\coprod_{\eta \in \overline{R}_{n+1}, n \geq 0} \eta\beta\alpha^{-n}K_0 \right)$$

where

$$\alpha = \begin{pmatrix} \pi^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{\pi} \end{pmatrix}, \quad \beta = \alpha s = \begin{pmatrix} 0 & 0 & \pi^{-1} \\ 0 & 1 & 0 \\ \overline{\pi} & 0 & 0 \end{pmatrix}$$

Proof. We have a Cartan decomposition $G = \coprod_{n \geq 0} K_0 \alpha^{-n} K_0$ and a Iwahori decomposition $K_0 = I \coprod IsI$. Let e be an edge in $\mathcal{T}$, say connecting u_0 to u_1. Then there exists a unique edge e' in the standard apartment of $\mathcal{T}$ such that the couple (e_{01}, e) is G-equivalent to (e_{01}, e'). Since e_{01} is fixed, they are I-equivalent. More generally, we denote the vertices in the standard apartment by $(v_n)_{n\in\mathbb{Z}}$, and let $e_{ij} = (v_i, v_j)$ for any i, j such that $|i - j| = 1$.

It follows that the edges $\{e_{2n,2n+1}, e_{2n+2,2n+1}\}_{n\in\mathbb{Z}}$ form a set of representatives for $I\backslash\mathcal{T}_1 \cong I\backslash G/I$. Moreover, $e_{2n,2n+1} = \alpha^n e_{0,1}$ and $e_{2n+2,2n+1} = \alpha^n e_{2,1} = \alpha^n \beta e_{0,1}$. It follows that a set of representatives for the double cosets $I\backslash G/I$ is given by $\{\alpha^n, \beta\alpha^n\}_{n\in\mathbb{Z}}$.

As $\alpha^{n+1}K_0 = \beta\alpha^{-n}K_0$ we see that in the decomposition

$$G = \left(\coprod_{n\geq 0} I\alpha^{-n}K_0\right) \coprod \left(\coprod_{n\geq 0} I\beta\alpha^{-n}K_0\right)$$

the RHS indeed covers G. To see that the union is disjoint, note that $I\alpha^{-n}K_0/K_0$ corresponds by action of I on $\alpha^{-n}v_0$ to vertices at distance $2n$ from v_0, which are also at distance $2n+1$ from v_1 (i.e. the geodesic from v_0 to them does not pass through v_1). However, we know that the stabilizer of $\alpha^{-n}v_0$ is $\alpha^{-n}K_0\alpha^n$. It follows that these vertices correspond bijectively to cosets $I/I \cap \alpha^{-n}K_0\alpha^n$.

Similarly, $I\beta\alpha^{-n}K_0/K_0$ corresponds by action of I on $\beta\alpha^{-n}v_0$ to vertices at distance $2(n+1)$ from v_0, which are also at distance $2n+1$ from v_1(i.e. the geodesic from v_0 to them passes through v_1). However, we know that the stabilizer of $\beta\alpha^{-n}v_0$ is $\beta\alpha^{-n}K_0\alpha^n\beta^{-1}$. It follows that these vertices correspond bijectively to cosets $I/I \cap \beta\alpha^{-n}K_0\alpha^n\beta^{-1}$.

Computation shows that

$$\alpha^{-n}K_0\alpha^n \cap I = \begin{pmatrix} \mathcal{O}_E & \pi^n\mathcal{O}_E & \pi^{2n}\mathcal{O}_E \\ \pi\mathcal{O}_E & \mathcal{O}_E & \pi^n\mathcal{O}_E \\ \pi\mathcal{O}_E & \pi\mathcal{O}_E & \mathcal{O}_E \end{pmatrix} \cap G$$

$$\beta\alpha^{-n}K_0\alpha^n\beta^{-1} \cap I = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \pi^{n+1}\mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \pi^{2n+2}\mathcal{O}_E & \pi^{n+1}\mathcal{O}_E & \mathcal{O}_E \end{pmatrix} \cap G$$

hence, the natural quotient maps $N_0 \hookrightarrow I$, $\quad \overline{N}_1 \hookrightarrow I$ induce isomorphisms

$$I/_{\alpha^{-n}K_0\alpha^n \cap I} \simeq N_0/_{N_{2n}}, \quad I/_{\beta\alpha^{-n}K_0\alpha^n\beta^{-1}\cap I} \simeq \overline{N}_1/_{\overline{N}_{2n+2}}$$

Consequently, the proposition follows. ☐

By the proof above, we see that in fact, considering only vertices of distance 2 from v_0, we get the following corollary:

Corollary 2.19. *Let $R_1, \overline{R}_1$ be as above. Then*

$$K_0\alpha^{-1}K_0 = I\alpha^{-1}K_0 \coprod I\beta K_0 = \left(\coprod_{\eta\in R_1} \eta\alpha^{-1}K_0\right) \coprod \left(\coprod_{\eta\in\overline{R}_1} \eta\beta K_0\right)$$

We also introduce a definition of "spheres" around the vertex v_0.

Definition 2.20. For any $0 \leq n \in \mathbb{Z}$, we let

$$S_n^0 = I\alpha^{-n}K_0 = \coprod_{\eta \in R_n} \eta\alpha^{-n}K_0, \quad S_{n+1}^1 = I\beta\alpha^{-n}K_0 = \coprod_{\eta \in \overline{R}_{n+1}} \eta\beta\alpha^{-n}K_0$$

Let $S_0 = S_0^0$, and for any $n \in \mathbb{N}$, let $S_n = S_n^0 \coprod S_n^1 = K_0\alpha^{-n}K_0$. Further, denote $B_n = \coprod_{i=0}^n S_i$.

Remark 2.21. We note that S_n/K_0 corresponds to the collection of vertices of distance $2n$ from v_0, and that B_n/K_0 corresponds to the collection of all vertices of type 0 of distance at most $2n$ from v_0.

2.3. Filtrations on the stabilizers

We will define certain decreasing filtrations on the stabilizers defined above - I, K_0 and K_1, by normal subgroups which are compact open in G.

We follow the construction described in (Schneider and Stuhler [21]), of filtrations on the stablilizers, and specialize it to our case, with $G = U_3(F)$.

It follows that we may define for each $e \geq 1$ the subgroups

$$I(e) = \begin{pmatrix} 1+\pi^e\mathcal{O}_E & \pi^{e-1}\mathcal{O}_E & \pi^{e-1}\mathcal{O}_E \\ \pi^e\mathcal{O}_E & 1+\pi^e\mathcal{O}_E & \pi^{e-1}\mathcal{O}_E \\ \pi^e\mathcal{O}_E & \pi^e\mathcal{O}_E & 1+\pi^e\mathcal{O}_E \end{pmatrix} \cap G$$

$$K_0(e) = \begin{pmatrix} 1+\pi^e\mathcal{O}_E & \pi^e\mathcal{O}_E & \pi^e\mathcal{O}_E \\ \pi^e\mathcal{O}_E & 1+\pi^e\mathcal{O}_E & \pi^e\mathcal{O}_E \\ \pi^e\mathcal{O}_E & \pi^e\mathcal{O}_E & 1+\pi^e\mathcal{O}_E \end{pmatrix} \cap G$$

$$K_1(e) = \begin{pmatrix} 1+\pi^e\mathcal{O}_E & \pi^{e-1}\mathcal{O}_E & \pi^{e-1}\mathcal{O}_E \\ \pi^e\mathcal{O}_E & 1+\pi^e\mathcal{O}_E & \pi^{e-1}\mathcal{O}_E \\ \pi^{e+1}\mathcal{O}_E & \pi^e\mathcal{O}_E & 1+\pi^e\mathcal{O}_E \end{pmatrix} \cap G$$

which are normal in I, K_0, K_1, respectively and compact open in G. In particular, we see that

$$K_0(1) = \begin{pmatrix} U_E^1 & \mathfrak{p}_E & \mathfrak{p}_E \\ \mathfrak{p}_E & U_E^1 & \mathfrak{p}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & U_E^1 \end{pmatrix} \cap G$$

$$I(1) = \begin{pmatrix} U_E^1 & \mathcal{O}_E & \mathcal{O}_E \\ \mathfrak{p}_E & U_E^1 & \mathcal{O}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & U_E^1 \end{pmatrix} \cap G$$

and

$$K_1(1) = \begin{pmatrix} U_E^1 & \mathcal{O}_E & \mathcal{O}_E \\ \mathfrak{p}_E & U_E^1 & \mathcal{O}_E \\ \mathfrak{p}_E^2 & \mathfrak{p}_E & U_E^1 \end{pmatrix} \cap G$$

where $U_E^1 = 1 + \mathfrak{p}_E$.

These filtrations play a role in the main theorem in (Schneider and Stuhler [21], II.3.1), which allows us to interpret a representation as the 0-th homology of a coefficient system.

Remark 2.22. One could also define the filtration corresponding to a facet σ by letting $U_\sigma^{(e)}$ be the stabilizer of all vertices which are at distance at most e from σ.

In particular, for $i = 0, 1$, $K_i(1)$ is the stabilizer of the star pointed at v_i , and $I(1)$ is the stabilizer of all triangles containing e_{01} as an edge. However, we will not use this description.

2.4. Lemmata on Finite fields

In what follows, p is a prime number, q is a power of p, and $\mathbb{F}_q$ is the unique field containing q elements.

$\mathbf{G}(\mathbb{F}_q) = \mathbf{U}_3(\mathbb{F}_q)$ is the unitary group in three variables over $\mathbb{F}_q$, i.e. $\mathbf{U}_3(\mathbb{F}_q) = \{g \in GL_3(\mathbb{F}_{q^2}) \mid {}^t\bar{g}sg = s\}$, and $\mathbb{F}_{q^2}^- = \{x \in \mathbb{F}_{q^2} \mid x + x^q = 0\}$.

We further let $\mathbf{B}(\mathbb{F}_q)$ be the Borel subgroup of upper triangular matrices, and $\mathbf{N}(\mathbb{F}_q)$ its unipotent radical. As before, we see that

$$\mathbf{N}(\mathbb{F}_q) = \left\{ n_{b,z} = \begin{pmatrix} 1 & b & z \\ 0 & 1 & -b^q \\ 0 & 0 & 1 \end{pmatrix} \mid b, z \in \mathbb{F}_{q^2}, \quad z + z^q + b^{q+1} = 0 \right\}$$

In addition, we consider $\mathbf{O}_3(\mathbb{F}_q) = \{g \in GL_3(\mathbb{F}_q) \mid^t gsg = s\}$, and its unipotent radical $\mathbf{N}'(\mathbb{F}_q)$, which is given by

$$\mathbf{N}'(\mathbb{F}_q) = \left\{ n_{b.z} = \begin{pmatrix} 1 & b & z \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} \mid b, z \in \mathbb{F}_q, \quad 2z + b^2 = 0 \right\}$$

Lemma 2.23. *For any q, one has*

$$|\mathbf{N}(\mathbb{F}_q)| = q^3, \quad |\mathbb{F}_{q^2}^-| = q, \quad |\mathbf{N}'(\mathbb{F}_q)| = q$$

Moreover, for any $0 \neq i \in \mathbb{F}_{q^2}^-$, $\mathbb{F}_{q^2}^- = \mathbb{F}_q \cdot i$, and if $p = 2$, then $\mathbb{F}_{q^2}^- = \mathbb{F}_q$.

Proof. Assume that $p \neq 2$, and let $\alpha \in \mathbb{F}_{q^2}^\times$ be a generator, then $\alpha^{q+1} \in \mathbb{F}_q^\times$ is not a square in $\mathbb{F}_q$, as $\frac{q+1}{2} \in \mathbb{Z}$ and

$$\left(\alpha^{\frac{q+1}{2}}\right)^{q-1} = \alpha^{\frac{q^2-1}{2}} = -1$$

so that $\alpha^{\frac{q+1}{2}} \notin \mathbb{F}_q$. Further, we have

$$\left(\alpha^{\frac{q+1}{2}}\right)^q = \alpha^{\frac{q(q+1)}{2}} = -\alpha^{\frac{q+1}{2}}$$

so that, denoting $i = \alpha^{\frac{q+1}{2}}$, we see that $\bar{i} = -i$. As $i \notin \mathbb{F}_q$, $\{1, i\}$ is a basis for $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$, and for any $b, z \in \mathbb{F}_{q^2}$ we may write $b = b_1 + b_2 i$, $z = z_1 + z_2 i$, with $b_1, b_2, z_1, z_2 \in \mathbb{F}_q$ so that $n_{b,z} \in \mathbf{N}(\mathbb{F}_q)$ if and only if

$$2z_1 = -(b_1^2 - b_2^2 i^2) \iff z_1 = -\frac{b_1^2 - b_2^2 i^2}{2}$$

Note that 2 is invertible in $\mathbb{F}_q$, so this makes sense. It follows, that given b, z_1 is uniquely determined, and z_2 can attain any value. It follows that $|\mathbf{N}(\mathbb{F}_q)| = q^3$ and $\left|\mathbb{F}_{q^2}^-\right| = q$.

Similarly, as 2 is invertible in $\mathbb{F}_q$, $\mathbf{N}'(\mathbb{F}_q) = \{n_{b,-b^2/2} \mid b \in \mathbb{F}_q\}$ is in bijection with $\mathbb{F}_q$, hence $|\mathbf{N}'(\mathbb{F}_q)| = q$.

If $p = 2$, we see that $z^q = -z \iff z^q = z$, hence $\mathbb{F}_{q^2}^- = \mathbb{F}_q$, and we still have $\left|\mathbb{F}_{q^2}^-\right| = |\mathbb{F}_q| = q$, and for some $\alpha \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$, $\{1, \alpha\}$ is a basis for $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$, and for any $b, z \in \mathbb{F}_{q^2}$ we may write $b = b_1 + b_2\alpha$, $z = z_1 + z_2\alpha$, with $b_1, b_2, z_1, z_2 \in \mathbb{F}_q$ so that $n_{b,z} \in \mathbf{N}(\mathbb{F}_q)$ if and only if

$$(\alpha + \alpha^q) \cdot z_2 = b_1^2 + b_2^2 \alpha^{q+1} b_2^2 + b_1 b_2(\alpha + \alpha^q) \iff z_2 = \frac{b_1^2 + b_2^2 \cdot \alpha^{q+1}}{\alpha + \alpha^q} + b_1 b_2$$

Since $\alpha \notin \mathbb{F}_q$, $\alpha \neq \alpha^q$, hence $\alpha + \alpha^q \neq 0$ is invertible, and z_2 is uniquely well defined for any b_1, b_2. z_1 is determined arbitrarily, hence again $|\mathbf{N}(\mathbb{F}_q)| = q^3$.

Also, $\mathbf{N}'(\mathbb{F}_q) = \{n_{b,z} \mid b = 0, z \in \mathbb{F}_q\} = \{n_{0,z} \mid z \in \mathbb{F}_q\}$ is again in bijection with $\mathbb{F}_q$, hence $|\mathbf{N}'(\mathbb{F}_q)| = q$. ☐

The following lemma is a consequence, which will be used in Section 7, in the course of the proof of the main theorem.

Lemma 2.24. *(a) Let $1 \neq \eta : \mathbb{F}_{q^2}^\times \to C$ be a character. Extend it to $\mathbb{F}_{q^2}$ by setting $\eta(0) = 0$. Then for any $0 \neq i \in \mathbb{F}_{q^2}^-$, one has*

$$\sum_{z \in \mathbb{F}_{q^2}^-} \eta(z) = \begin{cases} 0 & \eta\restriction_{\mathbb{F}_q^\times} \neq 1 \\ (q-1)\eta(i) & \eta\restriction_{\mathbb{F}_q^\times} = 1, \quad p \neq 2 \\ q-1 & \eta\restriction_{\mathbb{F}_q^\times} = 1, \quad p = 2 \end{cases}$$

and

$$\sum_{n_{b,z} \in \mathbf{N}(\mathbb{F}_q)} \eta(z) = \begin{cases} 0 & \eta\restriction_{\mathbb{F}_q^\times} \neq 1 \\ -q(q-1)\eta(i) & \eta\restriction_{\mathbb{F}_q^\times} = 1, \quad p \neq 2, \quad \eta \neq 1 \\ -q(q-1) & \eta\restriction_{\mathbb{F}_q^\times} = 1, \quad p = 2, \quad \eta \neq 1 \\ q^3 - 1 & \eta = 1 \end{cases}$$

(b) Let $p \neq 2$ and let$\eta : \mathbb{F}_q^\times \to C$ be a character. Extend it to $\mathbb{F}_q$ by setting $\eta(0) = 0$. Then

$$\sum_{b \in \mathbb{F}_q} \eta\left(-\frac{b^2}{2}\right) = \begin{cases} 0 & \eta \neq \varepsilon_q, 1 \\ \eta\left(-\frac{1}{2}\right) \cdot (q-1) & \eta = \varepsilon_q, 1 \end{cases}$$

where ε_q is the quadratic character on $\mathbb{F}_q$ defined by $\varepsilon_q(x) = \begin{cases} 1 & x \in \mathbb{F}_q^2 \\ -1 & x \notin \mathbb{F}_q^2 \end{cases}$.

Proof. (*a*) If $\eta \restriction_{\mathbb{F}_q^\times} \neq 1$, then there exists an $a \in \mathbb{F}_q^\times$ with $\eta(a) \neq 1$. However, for any $z \in \mathbb{F}_{q^2}^-$, one has $az \in \mathbb{F}_{q^2}^-$, so that

$$\sum_{z\in\mathbb{F}_{q^2}^-} \eta(z) = \sum_{z\in\mathbb{F}_{q^2}^-} \eta(az) = \sum_{z\in\mathbb{F}_{q^2}^-} \eta(a)\eta(z) = \eta(a) \sum_{z\in\mathbb{F}_{q^2}^-} \eta(z)$$

which shows that the sum vanishes.
If $\eta \restriction_{\mathbb{F}_q^\times} = 1$ and $p \neq 2$, then, as any $z \in \mathbb{F}_{q^2}^-$ is of the form $z = z_2 \cdot i$ (see Lemma 2.23) for $z_2 \in \mathbb{F}_q$, we see that $\eta(z) = \eta(i)$, hence

$$\sum_{z\in\mathbb{F}_{q^2}^-} \eta(z) = (q-1)\eta(i)$$

(recall that we have defined $\eta(0) = 0$).
If $\eta \restriction_{\mathbb{F}_q^\times} = 1$ and $p = 2$, then as $\mathbb{F}_{q^2}^- = \mathbb{F}_q$, we see that $\sum_{z\in\mathbb{F}_{q^2}^-} \eta(z) = q-1$. This settles the computation of the first sum.
Before we proceed, we note that for any $z \in \mathbb{F}_{q^2}$, as the norm map $\mathbb{F}_{q^2} \to \mathbb{F}_q$ is surjective, there exists $b \in \mathbb{F}_{q^2}$ such that $b\bar{b} = -(z+\bar{z})$.
Moreover, if $z \in \mathbb{F}_{q^2}^-$, it follows that $b = 0$ is the only possible value for b, and else, we have $\left|\mathbb{F}_{q^2}^1\right| = q+1$ different solutions for this equation.
Now, recalling that we have extended η so that $\eta(0) = 0$, we see that

$$\sum_{z\in\mathbb{F}_{q^2}} \eta(z) = \begin{cases} 0 & \eta \neq 1 \\ q^2-1 & \eta = 1 \end{cases}$$

Therefore $\sum_{z\notin\mathbb{F}_{q^2}^-} \eta(z) = -\sum_{z\in\mathbb{F}_{q^2}^-} \eta(z)$, when $\eta \neq 1$ is nontrivial. Thus we have

$$\sum_{n_{b,z}\in\mathbf{N}(\mathbb{F}_q)} \eta(z) = \sum_{z\in\mathbb{F}_{q^2}^-} \eta(z) + (q+1) \cdot \sum_{z\notin\mathbb{F}_{q^2}^-} \eta(z) =$$

$$= \begin{cases} 0 & \eta \restriction_{\mathbb{F}_q^\times} \neq 1 \\ -q(q-1)\eta(i) & \eta \restriction_{\mathbb{F}_q^\times} = 1, \quad p \neq 2, \quad \eta \neq 1 \\ -q(q-1) & \eta \restriction_{\mathbb{F}_q^\times} = 1, \quad p = 2, \quad \eta \neq 1 \\ q^3-1 & \eta = 1 \end{cases}$$

(*b*) Let $\alpha \in \mathbb{F}_q \backslash \mathbb{F}_q^2$ a non-square. Then any $x \in \mathbb{F}_q$ is either of the form $x = -\frac{b^2}{2}$ or $x = -\alpha \cdot \frac{b^2}{2}$ for some b. Moreover, for any $x \neq 0$ this is exactly a 2-to-one map, with $\pm b$ giving x, and for $x = 0$, we have $b = 0$ in both forms, hence if $\eta \neq 1$

$$0 = 2 \cdot \sum_{x \in \mathbb{F}_q} \eta(x) = \sum_{b \in \mathbb{F}_q} \eta\left(-\frac{b^2}{2}\right) + \sum_{b \in \mathbb{F}_q} \eta\left(-\alpha \cdot \frac{b^2}{2}\right) = (1 + \eta(\alpha)) \cdot \sum_{b \in \mathbb{F}_q} \eta\left(-\frac{b^2}{2}\right)$$

If $\eta \neq \varepsilon_q$, then there exists such α with $\eta(\alpha) \neq -1$, hence the sum vanishes. □

Corollary 2.25. *If E/F is unramified (so that $q = q_F^2$ and f is even), then $\mathcal{T}$ is bihomogeneous with degrees $q^{3/2} + 1$ and $q^{1/2} + 1$ for vertices of types 0,1 respectively. If E/F is ramified, then $\mathcal{T}$ is homogeneous of degree $q + 1$, where $q = q_E$ is the cardinality of the residue field of E.*

Proof. By the identification of vertices of the tree with $(G/K_0) \coprod (G/K_1)$, we see that the vertices of each type have the same degree. We note that $K_0K_1 = K_1 \coprod I\alpha^{-1}K_1$ and $K_1K_0 = K_0 \coprod I\beta K_0$.

Indeed, as each K_i acts transitively on the neighbours of v_i, it follows from the Iwasawa decomposition (Corollary 2.19). We thus see that these decompositions to left cosets of K_1 and K_0 correspond to the degrees of v_0, v_1, respectively.

In particular, as $K_1K_0 = K_0 \coprod \left(\coprod_{\eta \in \overline{R}_1} \eta\beta K_0\right)$, we see that the degree of type 1 vertices, d_1, is $\left|\overline{R}_1\right| + 1 = \left|\overline{N}_1/\overline{N}_2\right| + 1$.

To obtain the result for type 0 vertices, we must decompose $I\alpha^{-1}K_1$ to left cosets of K_1 in a similar manner, and obtain that ${}^{I}/_{I \cap \alpha^{-1}K_1\alpha} \simeq {}^{N_0}/_{N_1}$, hence the degree of type 0 vertices, d_0, is $|N_0/N_1| + 1$.

Assume first that E/F is unramified. In this case, we may take $\pi \in F$, so that $\overline{\pi} = \pi$. Here and in what follows, $\overline{n}_{b,z} = sn_{b,z}s$.

Consider first the homomorphism $\overline{N}_1 \to k_E^- = \{x \in k_E \mid x + x^{q_F} = 0\}$ defined by $\overline{n}_{b,z} \mapsto (\pi^{-1}z) \mod \pi$. This is well defined, since for $\overline{n}_{b,z} \in \overline{N}_1$, $z \in \pi\mathcal{O}_E$, whence $\pi^{-1}z \in \mathcal{O}_E$, and as $z + \overline{z} + b\overline{b} = 0$, with $b\overline{b} \in \pi^2\mathcal{O}_E$, we see that $\pi^{-1}z + \overline{\pi^{-1}z} \in \pi\mathcal{O}_E$, so that $(\pi^{-1}z) \mod \pi \in k_E^-$. This is also a homomorphism since

$$\overline{n}_{b,z} \cdot \overline{n}_{c,y} = \overline{n}_{b+c,y+z-b\overline{c}} \mapsto (\pi^{-1}(y + z)) \mod \pi$$

as $b\overline{c} \in \pi^2\mathcal{O}_E$ for all $\overline{n}_{b,z}, \overline{n}_{c,y} \in \overline{N}_1$. Further, , if $\overline{n}_{b,z} \in \overline{N}_2$ then $z \in \pi^2\mathcal{O}_E$, so that $\pi^{-1}z \mod \pi = 0$, showing that the map factors through $\overline{N}_2$. As this is precisely the kernel, we have an injective homomorphism $\overline{N}_1/\overline{N}_2 \to k_E^-$.

This map is bijective - indeed, if $\alpha \in k_E^-$, choose $a \in \mathcal{O}_F$ such that $a \mod \pi = Nm(\alpha)$. Then the polynomial $x^2 + a \in \mathcal{O}_F[x] \subset \mathcal{O}_E[x]$ has a root when reduced mod π, namely α. Therefore, by Hensel's Lemma, it has also a root $z \in \mathcal{O}_E$ such that $z \mod \pi = \alpha$. It follows that $z + \overline{z} = 0$, hence $z \in E^- = \{x \in E \mid x + \overline{x} = 0\}$. Now $\overline{n}_{0,\pi z} \in \overline{N}_1$ maps to α.

Now, by Lemma 2.23, $\left|k_E^-\right| = q_F = q^{1/2}$, showing that $d_1 = q^{1/2} + 1$.

For d_0, consider the reduction map $N_0 \to \mathbf{N}(k_F)$. It is a homomorphism with kernel N_1, hence $N_0/N_1 \simeq \mathbf{N}(k_F)$. As, by Lemma 2.23, $|\mathbf{N}(k_F)| = q^{3/2}$, we see that $d_0 = q^{3/2} + 1$.

Next, consider the case when E/F is ramified.

We now have a homomorphism $\overline{N}_1 \to k_E = k_F$ defined by $\overline{n}_{b,z} \mapsto (\pi^{-1}z) \mod \pi$. It is well defined, since for $\overline{n}_{b,z} \in \overline{N}_1$, $z \in \pi\mathcal{O}_E$, hence $\pi^{-1}z \in \mathcal{O}_E$. It is also a homomorphism with kernel $\overline{N}_2$, by the same reasoning as in the unramified case. Finally, if $\alpha \in k_E = k_F$, we may lift it to some $z \in F$ such that $z \mod \pi = \alpha$, and then $z = \overline{z}$. Choosing π such that $\pi^2 \in F$, we see that $\overline{\pi} = -\pi$, hence $\overline{\pi z} = -\pi z$, so that $\overline{n}_{0,\pi z} \in \overline{N}_1$ maps to α, showing surjectivity.

It follows that $d_1 = |k_E| + 1 = q + 1$.

For d_0, consider the reduction map $N_0 \to \mathbf{N}'(\mathbb{F}_q)$. It is a homomorphism with kernel N_1, hence $N_0/N_1 \simeq \mathbf{N}'(k_F)$. As, by Lemma 2.23, $|\mathbf{N}'(k_F)| = q$, we see that $d_0 = q + 1$. □

3. Representations of $U_3(F)$

3.1. $\mathbb{Q}_p$-algebraic representations of $U_3(F)$

As before, let C be a finite extension of $\mathbb{Q}_p$, containing E, such that

$$|Hom_{alg}(E, C)| = [E : \mathbb{Q}_p].$$

For $0 \le a, b \in \mathbb{Z}$, denote $S(a,b) = \mathrm{Sym}^a C^3 \otimes \mathrm{Sym}^b (C^3)^\star$. If $a, b \ge 1$ one has a natural contraction map $\iota_{a,b} : S(a,b) \to S(a-1, b-1)$ defined by

$$\iota_{a,b}\left(x_1 x_2 \cdots x_a \otimes y_1 y_2 \cdots y_b\right) = \sum_{i=1}^{a} \sum_{j=1}^{b} \langle x_i, y_j \rangle \left(x_1 x_2 \cdots \hat{x}_i \cdots x_a \otimes y_1 y_2 \cdots \hat{y}_j \cdots y_b\right)$$

Denote its kernel by $V(a,b)$. If $ab = 0$, we let $V(a,b) = S(a,b)$.

For $\tau \in Hom_{alg}(E, C)$, $d_\tau \in \mathbb{Z}$ and $0 \le a_\tau, b_\tau \in \mathbb{Z}$, we denote by $\rho(a_\tau, b_\tau, d_\tau)^\tau$ the irreducible algebraic representation of $U_3 \otimes_{F,\tau} C$ of highest weight

$$\chi_\tau : g = \mathrm{diag}(z_1, z_2, \overline{z}_1^{-1}) \mapsto \tau(z_1)^{a_\tau} \tau(\overline{z}_1)^{b_\tau} \cdot \tau(\det(g))^{d_\tau}$$

with respect to B.

We identify $\rho(a_\tau, b_\tau, d_\tau)^\tau$ with a representation of G on the C-vector space $V(a_\tau, b_\tau)$, as follows.

By choosing a basis $(x_{\tau,1}, x_{\tau,2}, x_{\tau,3})$ for $(C^3)^\star$, and a dual basis $(y_{\tau,1}, y_{\tau,2}, y_{\tau,3})$ for $C^3 = ((C^3)^\star)^\star$, we may identify $Sym^a C^3$ with the space of homogeneous polynomials of degree a in the $x_{\tau,k}$, and similarly identify $\mathrm{Sym}^b (C^3)^\star$ with the space of homogeneous polynomials of degree b in the $y_{\tau,k}$.

Under this identification, $S(a_\tau, b_\tau)$ is identified with the space of (a_τ, b_τ)-bihomogeneous polynomials in the $x_{\tau,k}, y_{\tau,k}$. Explicitly,

$$S(a_\tau, b_\tau) = \bigoplus_{\substack{i,j \in \mathbb{Z}^3_{\ge 0} \\ |i| = a_\tau, |j| = b_\tau}} C \cdot x_\tau^i y_\tau^j$$

where $x_\tau = (x_{\tau,1}, x_{\tau,2}, x_{\tau,3})$, $y_\tau = (y_{\tau,1}, y_{\tau,2}, y_{\tau,3})$, $i = (i_1, i_2, i_3), j = (j_1, j_2, j_3)$ are such that $|i| = i_1 + i_2 + i_3 = a_\tau$, $|j| = j_1 + j_2 + j_3 = b_\tau$, and we denote $x_\tau^i = x_{\tau,1}^{i_1} x_{\tau,2}^{i_2} x_{\tau,3}^{i_3}$ and $y_\tau^j = y_{\tau,1}^{j_1} y_{\tau,2}^{j_2} y_{\tau,3}^{j_3}$.

As $\langle x_{\tau,k}, y_{\tau,l} \rangle = \delta_{kl}$, the map $\iota_{a,b}$ now takes the following form

$$\iota_{a_\tau,b_\tau}\left(x_\tau^i y_\tau^j\right) = \iota_{a_\tau,b_\tau}\left(x_{\tau,1}^{i_1} x_{\tau,2}^{i_2} x_{\tau,3}^{i_3} \otimes y_{\tau,1}^{j_1} y_{\tau,2}^{j_2} y_{\tau,3}^{j_3}\right) =$$

$$= \sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{r=1}^{i_k}\sum_{s=1}^{j_l} \langle x_{\tau,k}, y_{\tau,l}\rangle \left(x_{\tau,k}^{-1} \cdot x_\tau^i \otimes y_{\tau,l}^{-1} \cdot y_\tau^j\right) =$$

$$= \sum_{k=1}^{3} i_k j_k x_{\tau,k}^{-1} y_{\tau,k}^{-1} \cdot x_\tau^i y_\tau^j = \left(\sum_{k=1}^{3} \frac{\partial^2}{\partial x_{\tau,k}\partial y_{\tau,k}}\right)\left(x_\tau^i y_\tau^j\right)$$

Therefore, we have

$$V(a_\tau, b_\tau) = \left\{ f(x_\tau, y_\tau) \in S(a_\tau, b_\tau) \mid \sum_{k=1}^{3} \frac{\partial^2 f}{\partial x_{\tau,k}\partial y_{\tau,k}} = 0 \right\}$$

This is the space of bihomogeneous polynomials of bidegree (a_τ, b_τ) with coefficients in C satisfying a certain differential equation. G acts on it as follows -

Any $\tau : E \hookrightarrow C$ induces an embedding, which we denote by the same letter $\tau : GL_3(E) \to GL_3(C)$, as there is no risk of confusion.

$GL_3(C)$ acts on C^3 naturally, hence for any τ we obtain a natural $G = U_3(F) \subset GL_3(E)$ action, which we denote by $(g, v) \mapsto \tau(g) \cdot v$.

This action also induces a natural right action on $(C^3)^\star$, namely $(v^\star \cdot \tau(g))(v) = v^\star(\tau(g) \cdot v)$ for any $v \in (C^3)^\star$, $v \in C^3$.

Therefore it induces an action on $S(a_\tau, b_\tau)$ defined by

$$(g \cdot f)(x_\tau, y_\tau) = \tau(\det(g))^{d_\tau} \cdot f(x_\tau \cdot \tau(g), \tau(g)^{-1} \cdot y_\tau)$$

Note that ι_{a_τ,b_τ} commutes with this action of G. Indeed, writing for simplicity g for $\tau(g)$, and x, y, a, b, d for $x_\tau, y_\tau, a_\tau, b_\tau, d_\tau$ we obtain

$$\iota_{a,b}(g \cdot f)(x, y) = \sum_{k=1}^{3} \frac{\partial^2 (g \cdot f)}{\partial x_k \partial y_k}(x, y) = \sum_{k=1}^{3} \frac{\partial}{\partial y_k}\left(\frac{\partial (g \cdot f)}{\partial x_k}\right)(x, y) =$$

$$= \det(g)^d \cdot \sum_{k=1}^{3} \frac{\partial}{\partial y_k}\left(\sum_{j=1}^{3} g_{kj} \cdot \frac{\partial f}{\partial x_j}(x \cdot g, g^{-1} y)\right) =$$

$$= \det(g)^d \cdot \sum_{k=1}^{3}\sum_{l=1}^{3} (g^{-1})_{lk} \left(\sum_{j=1}^{3} g_{kj} \cdot \frac{\partial^2 f}{\partial x_j \partial y_l}(x \cdot g, g^{-1} y)\right)$$

But $\sum_{k=1}^{3}(g^{-1})_{lk}g_{kj} = \delta_{lj}$, by definition, so that

$$\iota_{a,b}(g \cdot f)(x,y) = \det(g)^d \cdot \sum_{l=1}^{3} \frac{\partial^2 f}{\partial x_l \partial y_l}(x \cdot g, g^{-1}y) = g \cdot (\iota_{a,b}(f))(x,y)$$

Therefore $V(a_\tau, b_\tau) = \ker \iota_{a_\tau,b_\tau} \subseteq S(a_\tau, b_\tau)$ is a subrepresentation with the induced action.

This representation of G is a realization of $\rho(a_\tau, b_\tau, d_\tau)^\tau$, with $x_{\tau,1}^a y_{\tau,3}^b$ the highest weight vector, and $x_{\tau,3}^a y_{\tau,1}^b$ the lowest weight vector, with respect to B and M. If $v_\tau \in \rho(a_\tau, b_\tau, d_\tau)^\tau$ and $g \in G$, we will denote simply by gv_τ the action of g on v_τ.

Fix $\underline{a} = (a_\tau)_{\tau:E\hookrightarrow C}$, $\underline{b} = (b_\tau)_{\tau:E\hookrightarrow C}$ and $\underline{d} = (d_\tau)_{\tau:E\hookrightarrow C}$ such that $a_\tau, b_\tau \geq 0$.

Denote by $\rho_{\underline{a},\underline{b},\underline{d}}$ the representation of G on the underlying vector space

$$\rho(\underline{a}, \underline{b}, \underline{d}) = \bigotimes_{\tau:E\hookrightarrow C} \rho(a_\tau, b_\tau, d_\tau)^\tau$$

for which an element $g \in G$ acts componentwise.

In particular, for any $\bigotimes_{\tau:E\hookrightarrow C} v_\tau \in \rho(\underline{a}, \underline{b}, \underline{d})$,

$$\rho_{\underline{a},\underline{b},\underline{d}}(g) \left(\bigotimes_{\tau:E\hookrightarrow C} v_\tau \right) = \bigotimes_{\tau:E\hookrightarrow C} gv_\tau.$$

For $\underline{i} = (i_\tau)$, $\quad \underline{j} = (j_\tau)$ sequences of $i_\tau, j_\tau \in \mathbb{Z}_{\geq 0}^3$ with $|i_\tau| = a_\tau, |j_\tau| = b_\tau$, we denote

$$x^{\underline{i}} y^{\underline{j}} = \bigotimes_{\tau:E\hookrightarrow C} x_\tau^{i_\tau} y_\tau^{j_\tau}$$

The representations $\rho_{\underline{a},\underline{b},\underline{d}}$ are irreducible, and in fact exhaust all irreducible algebraic representations of $G = U_3(F)$.

This description of the irreducible algebraic representations of GL_3 is given e.g. in (Fulton and Harris [12]) over $\mathbb{C}$, and the consequence for representations of U_3 over C can be obtained, for example, by considering (Tits [26]).

Note that this description, up to a twist by a power of the determinant, exhausts all irreducible algebraic representations of G over C.

For any $\tau : E \hookrightarrow C$, we may consider an endomorphism $U_{a_\tau,b_\tau} \in End(S(a_\tau, b_\tau))$ whose action is described by

$$U_{a_\tau,b_\tau}(x_\tau^i y_\tau^j) = \tau(\pi)^{i_1 - j_1 + a_\tau} \cdot \tau(\overline{\pi})^{j_3 - i_3 + b_\tau} \cdot x_\tau^i y_\tau^j$$

for any $i, j \in \mathbb{Z}_{\geq 0}^3$ such that $|i| = a_\tau, |j| = b_\tau$.

Denote

$$U_{\underline{a},\underline{b},\underline{d}} = \bigotimes_{\tau:E\hookrightarrow C} U_{a_\tau,b_\tau,d_\tau} \tag{3.1}$$

Lemma 3.1. *There exists a unique function $\psi : G \to End_C(\rho(\underline{a}, \underline{b}, \underline{d}))$ supported in $K_0\alpha^{-1}K_0$ such that:*
(i) for all $k_1, k_2 \in K_0$, we have $\psi(k_1\alpha k_2) = \rho_{\underline{a},\underline{b},\underline{d}}(k_1) \circ \psi(\alpha^{-1}) \circ \rho_{\underline{a},\underline{b},\underline{d}}(k_2)$.
(ii) $\psi(\alpha^{-1}) = U_{\underline{a},\underline{b},\underline{d}}$.

Proof. Suppose there exist two such functions ψ_1, ψ_2. Then by (ii), $\psi_1(\alpha^{-1}) = \psi_2(\alpha^{-1})$, hence by (i), $\psi_1(g) = \psi_2(g)$ for all $g \in K_0\alpha^{-1}K_0$, hence the uniqueness.
For the existence, it suffices to show that ψ is well defined, i.e. that if $k_1\alpha^{-1}k_2 = \alpha^{-1}$ with $k_1, k_2 \in K_0$, then $\psi(k_1\alpha^{-1}k_2) = \psi(\alpha^{-1})$.
But for any $\underline{i} = (i_\tau)_{\tau:E\hookrightarrow C}$ and $\underline{j} = (j_\tau)_{\tau:E\hookrightarrow C}$ with $i_\tau, j_\tau \in \mathbb{Z}^3_{\geq 0}$ such that $|i_\tau| = a_\tau$ and $|j_\tau| = b_\tau$, one has

$$U_{\underline{a},\underline{b},\underline{d}} = \bigotimes_{\tau:E\hookrightarrow C} U_{a_\tau,b_\tau,d_\tau} = \left(\prod_{\tau:E\hookrightarrow C} \tau(\overline{\pi})^{a_\tau+d_\tau} \cdot \tau(\pi)^{b_\tau-d_\tau}\right) \cdot \rho_{\underline{a},\underline{b},\underline{d}}(\alpha^{-1}) \quad (3.2)$$

For brevity, denote

$$\pi^{\underline{a},\underline{b},\underline{d}} = \left(\prod_{\tau:E\hookrightarrow C} \tau(\overline{\pi})^{a_\tau+d_\tau} \cdot \tau(\pi)^{b_\tau-d_\tau}\right)$$

Therefore, if $k_1\alpha^{-1}k_2 = \alpha^{-1}$ for some $k_1, k_2 \in K_0$, then

$$\rho_{\underline{a},\underline{b},\underline{d}}(k_1) \circ \pi^{\underline{a},\underline{b},\underline{d}} \cdot \rho_{\underline{a},\underline{b},\underline{d}}(\alpha^{-1}) \circ \rho_{\underline{a},\underline{b},\underline{d}}(k_2) = \pi^{\underline{a},\underline{b},\underline{d}} \cdot \rho_{\underline{a},\underline{b},\underline{d}}(\alpha^{-1})$$

hence

$$\rho_{\underline{a},\underline{b},\underline{d}}(k_1) \circ \psi(\alpha^{-1}) \circ \rho_{\underline{a},\underline{b},\underline{d}}(k_2) = \psi(\alpha^{-1})$$

which finishes the proof. □

3.2. Compactly induced representations

In what follows R is either C or $\mathcal{O}_C$.

Definition 3.2. Let G be a topological group, and let H be a closed subgroup. Let (π, V) be a R-linear representation of H over a free R-module of finite rank V. We denote by $ind_H^G\pi$ or by $ind_H^G V$ the smooth compact induction of (π, V) from H to G. The space of the representation is

$$ind_H^G\pi = \left\{ f : G \to V \mid \begin{array}{c} f(hg) = \pi(h)f(g) \quad \forall h \in H \\ f \text{ has compact support mod } H, \quad f \text{ is smooth} \end{array} \right\}$$

and G acts on $ind_H^G\pi$ by right translation, i.e. $(gf)(x) = f(xg)$ for all $g, x \in G$.
If G is topological group, H is a closed subgroup, and (π, V) is an R-representation of H, we denote by $[g, v] \in ind_H^G\pi$ the function supported on the coset Hg^{-1} with value $v \in V$ at g^{-1}. Explicitly

$$[g, v](x) = \begin{cases} \pi(h)(v) & x = hg^{-1} \\ 0 & x \notin Hg \end{cases} \quad (3.3)$$

Note that the following identities hold

$$\forall g, g_1, g_2 \in G \quad g_1[g_2, v] = [g_1 g_2, v], \quad \forall g \in G, \forall h \in H \quad [gh, v] = [g, \pi(h)(v)]$$

As the type 0 vertices of the Bruhat-Tits tree correspond to left cosets of K_0 in G, we have an isomorphism of G-sets between $\mathcal{T}_0^0$ and $\{[g,1]\}_{g \in G/K_0}$, and we will often consider the functions $[g,1] \in ind_{K_0}^G \mathbf{1}$, where $\mathbf{1}$ is the trivial representation, as the type 0 vertex in the tree, $g^{-1}v_0$.

We recall also the following result, giving a basis for the $R[G]$-module $ind_H^G \pi$ when H is open. (see Barthel et al. [4]).

Proposition 3.3. *Let H be an open subgroup of G. Let $\mathcal{B}$ be a basis of (π, V) over R and $\mathcal{G}$ a system of representatives for the left cosets G/H. Then the family of functions $\mathcal{I} = \{[g, v] \mid g \in \mathcal{G}, v \in \mathcal{B}\}$ is a basis for $ind_H^G \pi$.*

3.3. Locally algebraic principal series representations

Let $M = \mathbf{M}(F)$ be the standard maximal torus of B consisting of diagonal matrices.

Definition 3.4. Let $\chi : M \to C^\times$ be a C-character of M inflated to B. The smooth *principal series representation* corresponding to χ is

$$ind_B^G(\chi) = \left\{ f : G \to C \mid \begin{array}{c} \exists U_f \text{ open s.t. } f(bgk) = \chi(b) f(g) \\ \forall g \in G, \quad b \in B, \quad k \in U_f \end{array} \right\}$$

with the group G acting by right translations, namely $(gf)(x) = f(xg)$ for all $x, g \in G$ and $f \in ind_B^G(\chi)$.

Note that the maximal torus $M = \mathbf{M}(F)$ of B is of the form

$$M = \left\{ m_{t,s} := \begin{pmatrix} t & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \bar{t}^{-1} \end{pmatrix} \mid t \in E^\times, s \in E^1 \right\} \simeq E^\times \times E^1$$

Therefore, any smooth character $\chi_M : M \to C^\times$ is of the form $\chi_M = \chi \otimes \chi_1$, where $\chi : E^\times \to C$ and $\chi_1 : E^1 \to C$ are smooth characters, i.e. $\chi_M(m_{t,s}) = \chi(t)\chi_1(s)$.

Remark 3.5. Note that in this case the induction is compact by the Iwasawa decomposition, $G = BK_0$, showing that any function $f \in ind_B^G(\chi \otimes \chi_1)$ is compactly supported modulo B.

The representations we shall be interested in are the locally algebraic ones, meaning that every vector has a neighbourhood in which the action is polynomial. To be precise[1],

[1] In fact, the usual definition of a locally algebraic representation is using only condition 2, see e.g. Emerton [10]. However, the following theorem is from the appendix of (Schneider et al. [23]), so we follow the definition given there. In the case of completed H^1 of modular or Shimura curves (by an argument of Emerton), or in the case of completed H^0 of definite unitary groups, condition 2 implies condition 1.

Definition 3.6. A C-representation (π, V) of G is called *locally algebraic* if:

1. The restriction of π to any compact open subgroup K of G is an algebraic direct sum of finite dimensional irreducible representations of K.
2. For any vector $v \in V$, there exists a compact open subgroup K_v in G, and a finite dimensional subspace U of V, containing v, such that K_v leaves U invariant, and operates on U via restriction to K_v of a finite dimensional algebraic representation of $\mathbf{G}$.

Definition 3.7. The locally algebraic C-representation of G, $V_{sm} \otimes V_{alg}$, where V_{sm} is a smooth representation, and V_{alg} is an algebraic representation, is called a *locally algebraic principal series representation* if V_{sm} is a smooth principal series representation.

Recall the following result (see Appendix of (Schneider et al. [23]), Thm 1) about irreducibility of locally algebraic representations -

Theorem 3.8. *Every irreducible locally algebraic representation π of G is the tensor product $\pi = \pi_{alg} \otimes \pi_{sm}$ of an irreducible algebraic representation π_{alg} of G and a smooth irreducible algebraic representation π_{sm} of G. Conversely, the tensor product $\pi_{alg} \otimes \pi_{sm}$ of an irreducible algebraic representation π_{alg} of G and an irreducible smooth representation π_{sm} of G, is an irreducible locally algebraic representation of G.*

As in this work, we are interested in irreducible locally algebraic representations, we will only consider representations of the form $\pi_{alg} \otimes \pi_{sm}$. Furthermore, the cases where π_{sm} is essentially discrete series or supercuspidal are known (see Sorensen [25]), hence we consider only the cases where π_{sm} is an irreducible smooth principal series representation as above.

We remark that the Breuil-Schneider conjecture also deals with indecomposable reducible representations, however we will not address this case presently.

Thus, by the above classification of irreducible algebraic representations, we are interested in representations of the form

$$(\pi, V) = Ind_B^G(\chi \otimes \chi_1) \otimes \rho_{\underline{a},\underline{b},\underline{d}}$$

where $\chi : E^\times \to C$, $\chi_1 : E^1 \to C$ are smooth characters and $\underline{a}, \underline{b} \in \mathbb{Z}_{\geq 0}^{Hom_{alg}(E,C)}$.

3.4. Spherical Hecke algebras

Let R be either C or $\mathcal{O}_C$, let K be an open compact subgroup of G, and let ρ be a continuous R-linear representation of K over a free R-module V_ρ of finite rank. The Hecke algebra $\mathcal{H}_\rho(K, G)$ associated to K and ρ is the R-algebra defined by

$$\mathcal{H}_\rho(K, G) = End_{R[G]}(ind_K^G \rho).$$

By Frobenius reciprocity for compact induction, for any R-representation π of G, one has

$$Hom_{R[G]}(ind_K^G \rho, \pi) \simeq Hom_{R[K]}(\rho, \pi \mid_K).$$

Thus, we can interpret $\mathcal{H}_\rho(K,G)$ as the convolution algebra

$$\mathcal{H}_K(\rho) = \left\{ \psi : G \to End_R(V_\rho) \mid \begin{array}{c} \psi(k_1 g k_2) = \rho(k_1) \circ \psi(g) \circ \rho(k_2) \quad \forall g \in G, k_1, k_2 \in K \\ \psi \text{ has compact support} \end{array} \right\}.$$

The convolution operation is defined for any $\psi_1, \psi_2 \in \mathcal{H}_K(\rho)$ and any $g \in G$ by

$$(\psi_1 * \psi_2)(g) = \sum_{xK \in G/K} \psi_1(x)\psi_2(x^{-1}g)$$

It admits a unit element ψ_e, supported on K, attaining the identity at 1, i.e.

$$\varphi_e(g) = \begin{cases} \rho(g) & g \in K \\ 0 & g \notin K \end{cases}$$

Then the bilinear map

$$\begin{array}{rcl} \mathcal{H}_K(\rho) \times ind_K^G \rho & \to & ind_K^G \rho \\ (\psi, f) & \mapsto & \langle \psi, f \rangle (g) := \sum_{xK \in G/K} \psi(x)(f(x^{-1}g)) \end{array}$$

gives $ind_K^G \rho$ a structure of a left $\mathcal{H}_K(\rho)$-module, which commutes with the action of G.

Lemma 3.9. *The map*

$$\begin{array}{rcl} \mathcal{H}_K(\rho) & \to & \mathcal{H}_\rho(K,G) \\ \psi & \mapsto & T_\psi(f) := \langle \psi, f \rangle \end{array}$$

is an isomorphism of R-algebras. In particular, if $g \in G$ and $v \in V_\rho$, the action of T_ψ on $[g,v]$ is given by

$$T_\psi([g,v]) = \sum_{xK \in G/K} [gx, \psi(x^{-1})(v)] \tag{3.4}$$

Proof. This is straight forward and well-known. See e.g. (De Ieso [9] Lemma 2.4). □

We recall further that when ρ is the restriction to K of a continuous representation of G, there exists an injective homomorphism of C-algebras (Schneider et al. [24])-

$$\begin{array}{rcl} \iota_\rho : \mathcal{H}_K(C) & \to & \mathcal{H}_K(\rho) \\ \varphi & \mapsto & (\varphi \cdot \rho)(g) = \varphi(g)\rho(g) \end{array}$$

where C is the trivial representation of G on C. This homomorphism is in fact bijective for certain irreducible locally $\mathbb{Q}_p$-analytic representations ρ, in the sense of (Schneider and Teitelbaum [22]) , by the following Lemma (see De Ieso [9] 2.5)

Lemma 3.10. *Let $\mathfrak{g} = Lie(G)$. Then if the $\mathfrak{g} \otimes_{\mathbb{Q}_p} C$-module V_ρ is absolutely irreducible, the map ι_ρ is bijective.*

Recall that by Lemma 3.1, we have constructed $\psi \in \mathcal{H}_{K_0}(\rho_{\underline{a},\underline{b}})$ for any $\underline{a}, \underline{b} \in \mathbb{Z}_{\geq 0}^{Hom_{alg}(E,C)}$. By Lemma 3.9, it corresponds to a Hecke operator $T \in \mathcal{H}_\rho(K_0, G)$, whose action on the elements $[g, v]$ are given by the formula (3.4).

Lemma 3.11. *There is a C-algebra isomorphism*

$$\mathcal{H}_{\rho_{\underline{a},\underline{b},\underline{d}}}(K_0, G) \simeq C[T]$$

Proof. The space $V_{\rho_{\underline{a},\underline{b},\underline{d}}}$ is an absolutely irreducible $\mathfrak{g} \otimes_{\mathbb{Q}_p} C$-module, hence $\iota_{\rho_{\underline{a},\underline{b},\underline{d}}}$ is an isomorphism of C-algebras. Lemma 3.9 shows that there exists a unique morphism of C-algebras $u_{\rho_{\underline{a},\underline{b},\underline{d}}} : \mathcal{H}_C(K_0, G) \to \mathcal{H}_{\rho_{\underline{a},\underline{b},\underline{d}}}(K_0, G)$ making the following diagram commute

$$\begin{array}{ccc} \mathcal{H}_{K_0}(C) & \xrightarrow{\sim} & \mathcal{H}_C(K_0, G) \\ \downarrow{\scriptstyle \iota_{\rho_{\underline{a},\underline{b},\underline{d}}}} & & \downarrow{\scriptstyle u_{\rho_{\underline{a},\underline{b},\underline{d}}}} \\ \mathcal{H}_{K_0}(\rho_{\underline{a},\underline{b},\underline{d}}) & \xrightarrow{\sim} & \mathcal{H}_{\rho_{\underline{a},\underline{b},\underline{d}}}(K_0, G) \end{array} \tag{3.5}$$

By construction, this morphism is an isomorphism of C-algebras. Denote by $T_1 \in \mathcal{H}_C(K_0, G)$ the element corresponding to $\mathbf{1}_{K_0\alpha^{-1}K_0} \in \mathcal{H}_{K_0}(C)$ by Frobenius reciprocity.

If $\varphi \in \mathcal{H}_{K_0}(C)$, then as it has compact support, by the Cartan decomposition (Proposition 2.15), it is supported on $\coprod_{i=0}^n K_0\alpha^{-i}K_0$ for some integer n. As φ is K_0-biinvariant (recall that C is the trivial representation), its restriction to each $S_i = K_0\alpha^{-i}K_0$ is constant, hence we may write $\varphi = \sum_{i=0}^n \varphi_i \cdot \mathbf{1}_{K_0\alpha^{-i}K_0}$. Let $T_i \in \mathcal{H}_C(K_0, G)$ be the operator corresponding to $\mathbf{1}_{K_0\alpha^{-i}K_0}$ by Frobenius reciprocity. Then we see that the T_n's span $\mathcal{H}_C(K_0, G)$ over C. Geometrically, T_n is the operator associating to a vertex v of type 0 the sum of the vertices of distance $2n$ from v: this is because

$$\mathbf{1}_{K_0\alpha^{-n}K_0} = \sum_{K_0x \in K_0 \backslash K_0\alpha^{-n}K_0} \mathbf{1}_{K_0x} =$$

$$= \sum_{K_0x \in K_0 \backslash K_0\alpha^{-n}K_0} [x^{-1}, 1] = \sum_{K_0x \in K_0 \backslash K_0\alpha^{-n}K_0} x^{-1} \cdot [1, 1]$$

and then the $x^{-1}v_0$ are all distinct and give all vertices $v' \in \mathcal{T}_0^0$ such that v' is K_0-equivalent to v_{2n}. This means that (v_0, v') is equivalent to (v_0, v_{2n}), which is precisely our assertion. From the geometrical description of T_n, one gets directly, using Corollary 2.25, that

$$T_1^2 = \begin{cases} T_2 + (q^{1/2} - 1)T_1 + (q^{3/2} + 1)q^{1/2} & E/F \quad \textit{unramified} \\ T_2 + (q-1)T_1 + (q+1)q & E/F \quad \textit{ramified} \end{cases}$$

Explicitly, let d_0, d_1 be the degrees of vertices of types 0,1 respectively.

If z is a vertex such that $d(z,v) = 4$, then there is a unique vertex w lying on the geodesic from v to z with $d(z,w) = 2 = d(v,w)$. Therefore, when applying T_1^2, each of them is counted once.

If z is a vertex such that $d(z,v) = 2$, then there is a unique vertex u lying on the geodesic from v to z, which is a common neighbour. This u has $d_1 - 2$ other neighbours whcih are at distance 2 from v, hence whem applying T_1^2, it is counted $d_1 - 2$ times.

Finally, v is counted once from every vertex of distance 2 from it. As there are $d_0 \cdot (d_1 - 1)$ such vertices, it is counted $d_0 \cdot (d_1 - 1)$ times.

Since T_1^2 yields only vertices at distances $0, 2, 4$ from v, these are all the possibilities, and we get

$$T_1^2 = T_2 + (d_1 - 2)T_1 + d_0 \cdot (d_1 - 1)$$

When we plug in the degrees in each of the cases, we get the answer.

In any case, it follows that $T_2 \in C[T_1]$. Furthermore, for any $n \geq 3$, we see that

$$T_1 T_{n-1} = \begin{cases} T_n + (q^{1/2} - 1)T_{n-1} + q^2 T_{n-2} & E/F \quad unramified \\ T_n + (q-1)T_{n-1} + q^2 T_{n-2} & E/F \quad ramified \end{cases}$$

showing that if $T_{n-1} \in C[T_1]$, then also $T_n \in C[T_1]$.

Again, if we consider some vertex at distance $2n$ from v, there is a unique vertex at distance $2(n-1)$ from v between them, which accounts for it when applying $T_1 T_{n-1}$.

If we consider a vertex at distance $2(n-1)$ from v, there are $d_1 - 2$ vertices of the same distance sharing a common neighbour with it, hence it is counted $d_1 - 2$ times when applying $T_1 T_{n-1}$.

Finally, if we have a vertex at distance $2(n-2)$ from v, there are $(d_0 - 1)(d_1 - 1)$ vertices at distance $2(n-1)$ from v, having this vertex on the geodesic, thus each such vertex is counted $(d_0 - 1)(d_1 - 1)$ times when applying $T_1 T_{n-1}$.

When we plug in the degrees in each of the cases, we obtain the result.

It follows that $\mathcal{H}_C(K_0, G) \simeq C[T_1]$.

As $u_{\rho_{\underline{a},\underline{b},\underline{d}}}(T_1) = (\pi^{\underline{a},\underline{b},\underline{d}})^{-1} \cdot T$, it follows that $\mathcal{H}_{\rho_{\underline{a},\underline{b},\underline{d}}}(K_0, G) \simeq C[T]$. □

Finally, as described by Kato (Kato [17], Thm 3.2), we have an intimate connection between the Hecke algebra and the irreducible smooth principal series representations.

Theorem 3.12. *Let $\chi : M \to C^\times$ be a smooth unramified character. Assume $\phi \in ind_B^G \chi$ is supported on BK_0 and given there by $\phi(bk) = \chi(b)$. Then $\phi \in \left(ind_B^G \chi\right)^{K_0}$.*

Assume $ind_B^G\chi$ *is generated by* ϕ*. Define* $j : ind_{K_0}^G C \to ind_B^G\chi$ *by* $j(f) = F_f$*, where*

$$F_f(g) := \int_G \phi(gh^{-1})f(h)dh$$

Note: (i) $F_f(bg) = \chi(b)F_f(g)$
(ii) $F_f(g) = \sum_{h\in K_0\backslash G}\phi(gh^{-1})f(h) = (f*\phi)(g)$
Then j *induces an isomorphism of* G*-representations*

$$ind_B^G\chi \simeq \frac{ind_{K_0}^G C}{(T_1 - \alpha_\chi(T_1))\cdot ind_{K_0}^G C}$$

where $\alpha_\chi : \mathcal{H}_C(K_0, G) \to C$ *is an algebra homomorphism of the Hecke algebra induced by* χ*, and* $\mathcal{H}_C(K_0, G) \simeq C[T_1]$*.*

In particular, when $ind_B^G\chi$ is irreducible, this holds. Thus, we obtain the following corollary.

Corollary 3.13. *Let* $\chi : E^\times \to C^\times$ *be a smooth unramified character, and let* $\underline{a}, \underline{b} \in \mathbb{Z}_{\geq 0}^{Hom_{alg}(E,C)}$*,* $\underline{d} \in \mathbb{Z}^{Hom_{alg}(E,C)}$*. Then if* $ind_B^G\chi$ *is irreducible, one has*

$$ind_B^G\chi \otimes \rho_{\underline{a},\underline{b},\underline{d}} \simeq \frac{ind_{K_0}^G \rho_{\underline{a},\underline{b},\underline{d}}}{(T - \alpha_\chi(T))\cdot ind_{K_0}^G \rho_{\underline{a},\underline{b},\underline{d}}}$$

Proof. As $\rho_{\underline{a},\underline{b},\underline{d}}$ is a finite dimensional representation, tensoring with it is an exact functor, thus

$$ind_B^G\chi \otimes \rho_{\underline{a},\underline{b},\underline{d}} \simeq \frac{\left(ind_{K_0}^G C\right) \otimes \rho_{\underline{a},\underline{b},\underline{d}}}{\left((T_1 - \alpha_\chi(T_1))\cdot ind_{K_0}^G C\right) \otimes \rho_{\underline{a},\underline{b},\underline{d}}}$$

However, by the commutative diagram (3.5), we see that

$$\frac{\left(ind_{K_0}^G C\right) \otimes \rho_{\underline{a},\underline{b},\underline{d}}}{\left((T_1 - \alpha_\chi(T_1))\cdot ind_{K_0}^G C\right) \otimes \rho_{\underline{a},\underline{b},\underline{d}}} \simeq \frac{ind_{K_0}^G \rho_{\underline{a},\underline{b},\underline{d}}}{\left(u_{\rho_{\underline{a},\underline{b},\underline{d}}}(T_1) - \alpha_\chi(u_{\rho_{\underline{a},\underline{b},\underline{d}}}(T_1))\right)\cdot ind_{K_0}^G \rho_{\underline{a},\underline{b},\underline{d}}}$$

But, as $u_{\rho_{\underline{a},\underline{b},\underline{d}}}(T_1) = (\pi^{\underline{a},\underline{b}})^{-1}\cdot T$, the result follows. □

In order to better understand the values of $\alpha_\chi(T_1)$, we present the universal principal series.

3.5. The universal principal series

In this section, and in this section alone, $R = C[t, t^{-1}]$ where t is a variable, and let $\chi : B \to R^\times$ be the smooth character

$$\chi\begin{pmatrix} a & * & * \\ 0 & * & * \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} = t^{val_E(a)}$$

Note that these are not R and χ as they appear in previous sections.

The universal unramified principal series representation is

$$\pi(\chi) = ind_B^G \chi$$

Here $val_E : E^\times \to \mathbb{Z}$ is the normalized valuation on E such that $val_E(\pi_E) = 1$. Its specialization under $t \mapsto \lambda \in C^\times$ is $\pi(\chi_\lambda)$, the unramified principal series representation.

For any $f \in \pi(\chi)$, one may associate a locally-constant R-valued function $\overline{f}$ on N by the rule $\overline{f}(n) = f(sn)$.

Lemma 3.14. *(i) This procedure identifies $\pi(\chi)$ with the space of locally constant functions on N satisfying*

$$\overline{f}(n_{b,z}) = const \cdot t^{-val_E(z)}$$

for all z large enough.

(ii) $\pi(\chi)$ is free of countable rank over R.

Proof. (*i*) We have

$$s \cdot \begin{pmatrix} 1 & b & z \\ 0 & 1 & -\overline{b} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \overline{z}^{-1} & -z^{-1}b & 1 \\ 0 & -z^{-1}\overline{z} & -\overline{b} \\ 0 & 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\overline{z}^{-1}\overline{b} & 1 & 0 \\ z^{-1} & z^{-1}b & 1 \end{pmatrix}$$

hence for large z, $z^{-1}b \to 0$, so one has $\overline{f}(n_{b,z}) = t^{-val_E(z)} \cdot f(1)$. By Bruhat decomposition, this is reversible.

Indeed, if $F \in C^\infty(N)$ is a locally constant function satisfying $F(n_{b,z}) = c \cdot t^{-val_E(z)}$ for all z large enough, then we may define

$$f_F(bsn) = \chi(b) \cdot F(n) \quad \forall b \in B, \quad n \in N$$

and

$$f_F(b) = \chi(b) \cdot c \quad \forall b \in B$$

The function f_F is well defined, since $G = BsN \coprod B$, with unique representatives, by the Bruhat decomposition.

Moreover, it is locally constant, as for $g \in BsN$ it follows from the fact that F is locally constant, and by the above equation

$$f_F \begin{pmatrix} 1 & 0 & 0 \\ -\overline{z}^{-1}\overline{b} & 1 & 0 \\ z^{-1} & z^{-1}b & 1 \end{pmatrix} = \chi^{-1} \begin{pmatrix} \overline{z}^{-1} & -z^{-1}b & 1 \\ 0 & -z^{-1}\overline{z} & -\overline{b} \\ 0 & 0 & z \end{pmatrix} \cdot F(n_{b,z})$$

so that for all z large enough,

$$f_F \begin{pmatrix} 1 & 0 & 0 \\ -\overline{z}^{-1}\overline{b} & 1 & 0 \\ z^{-1} & z^{-1}b & 1 \end{pmatrix} = t^{val_E(z)} \cdot c \cdot t^{-val_E(z)} = c = f_F(1)$$

This gives us an open compact neighbourhood of 1, such that f_F is constant there, and by translation, it shows that f_F is locally constant on B.
Finally, we note that $\overline{f_F}(n) = f_F(sn) = F(n)$ for all $n \in N$, and vice versa,

$$f_{\overline{f}}(bsn) = \chi(b) \cdot \overline{f}(n) = \chi(b) \cdot f(sn) = f(bsn), \quad f_{\overline{f}}(b) = \chi(b) \cdot f(1) = f(b)$$

showing that these are inverses.

Note that the functions $\overline{f}$ of compact support correspond to f vanishing on a neighbourhood of B.
(ii) In this model, we see that $\pi(\chi) \cong C_c^\infty(N, R) \oplus R\overline{f}_0$, where

$$\overline{f}_0(n) = \begin{cases} 1 & n \in \mathbf{N}(\mathcal{O}_E) \\ t^{-val_E(z)} & n \notin \mathbf{N}(\mathcal{O}_E) \end{cases}$$

The space of locally constant function of compact support is easily seen to be free of countable rank. □

3.5.1. Normed isotropic lines

Recall that $B\backslash G$ is identified with the space $\mathcal{F}$ of isotropic lines in $V = E^3$. We denote such a line by ξ.
The identification of $B\backslash G$ with $\mathcal{F}$ sends Bg to $\xi(g)$ where

$$\xi(g) = E \cdot g^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Let $B_0 = \ker(\chi)$. We may similarly identify $B_0\backslash G$ with the space $\widehat{\mathcal{F}}$ of equivalence classes of pairs

$$\hat{\xi} = [\xi; M]$$

where ξ is an isotropic line, and $M \subset \xi$ is a lattice.
The identification with $B_0\backslash G$ sends $B_0 g$ to $\widehat{\xi}(g) = [\xi(g); M(g)]$ where

$$M(g) = \mathcal{O}_E \cdot g^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We call $\hat{\xi}$ a *normed isotropic line.*
Indeed, we have a transitive action of G on the space of normed isotropic lines - namely $g[\xi; M] = [g\xi; gM]$.
Consider the normed isotropic line $\hat{\xi}(1) = [e_1; \mathcal{O}_E e_1] = [\xi_0; M_0]$. Then g stablilizes $\hat{\xi}(1)$ if and only if $g \in B$, and g stabilizes $\mathcal{O}_E e_1$ - this condition is equivalent to $g_{11} \in \mathcal{O}_E^\times$. Hence the stabilizer is B_0.

The action of G on $\widehat{\mathcal{F}}$ corresponds to the action on $B_0\backslash G$ by right translation

$$g\hat{\xi}(g') = g \cdot (g')^{-1}\hat{\xi}(1) = \hat{\xi}(g'g^{-1})$$

But $B_0\backslash G$ carries also a commuting action of B on $B_0\backslash G$ by left translation, which on $\widehat{\mathcal{F}}$ will be denoted $b * \hat{\xi}$, i.e.

$$b * \hat{\xi}(g) = \hat{\xi}(bg)$$

It is well defined since $B_0 \trianglelefteq B$. If $b = \begin{pmatrix} z & * & * \\ 0 & y & * \\ 0 & 0 & \overline{z}^{-1} \end{pmatrix}$, then this action is given explicitly by

$$b * [\xi; M] = [\xi; z^{-1}M]$$

3.5.2. The embedding of $ind_{K_0}^G C$ in $\pi(\chi)$

We shall now give a geometric interpretation for $\pi(\chi)$. Recall that $R = C[t, t^{-1}]$. Let $C_\chi(\widehat{\mathcal{F}})$ be the space of locally constant R-valued functions f on $\widehat{\mathcal{F}}$ which are B-equivariant in the sense that

$$f(b * \hat{\xi}) = \chi(b) \cdot f(\hat{\xi})$$

for all $b \in B$. Explicitly, taking $g = \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{\pi}^{-1} \end{pmatrix} = \alpha^{-1}$, we see that

$$f([\xi; \pi^{-1}M]) = t \cdot f([\xi; M])$$

We let G act on $C_\chi(\widehat{\mathcal{F}})$ via its action on $\widehat{\mathcal{F}}$

$$gf(\hat{\xi}) = f(g^{-1}\hat{\xi}).$$

We may now identify $\pi(\chi)$ G-equivariantly with $C_\chi(\widehat{\mathcal{F}})$, because a locally constant function $f : G \to R$ satisfying

$$f(bg) = \chi(b)f(g)$$

gives rise to a function $\overline{f} : \widehat{\mathcal{F}} \to R$ satisfying $\overline{f}(b * \hat{\xi}) = \chi(b)\overline{f}(\hat{\xi})$ if we let $\overline{f}(\hat{\xi}(g)) = f(g)$ and vice versa.

Using this interpretation, we now define $\iota : ind_{K_0}^G C \to \pi(\chi)$, or what is the same, $\iota : ind_{K_0}^G C \to C_\chi(\widehat{\mathcal{F}})$.

It is enough to define $\iota(1_v)$ for some $v \in \mathcal{T}_0^0$. Let ξ be an isotropic line. Let L be a standard lattice ($L = L^\sharp$) such that $v = [L]$. Then L determines a normed isotropic line $\xi_L = [\xi; M]$ over ξ by letting $M = L \cap \xi$.

The function $\iota(1_v)$ is uniquely defined on the fiber above ξ by the requirement that

$$\iota(1_v)(\xi_L) = 1$$

When v_0 is the standard lattice, this definition means, group theoretically, that $\iota(1_{K_0})$ is defined on $BK = G$ by $\iota(1_{K_0})(bk) = \chi(b)$ for $b \in B$ and $k \in K$.

3.5.3. Theorem

(*i*) The map ι is G-equivariant and, denoting $q = q_E$,

$$\iota(T_1\phi) = \begin{cases} (t\cdot q^2 + t^{-1} + (q^{1/2}-1))\iota(\phi) & E/F \quad unramified \\ (t\cdot q^2 + t^{-1} + (q-1))\iota(\phi) & E/F \quad ramified \end{cases}$$

(*ii*) The map ι induces by specializing $t \mapsto \lambda$, $\iota_\lambda : ind_{K_0}^G C \to \pi(\chi_\lambda)$ is the map j described in Theorem 3.12, and when $ind_B^G \chi_\lambda$ is irreducible, induces an isomorphism

$$ind_B^G \chi_\lambda \simeq \frac{ind_{K_0}^G C}{(T_1 - \lambda^{-1} - \lambda \cdot q^2 - (q^\epsilon - 1)) \cdot ind_{K_0}^G C}$$

with $\epsilon = 1/2$ if E/F is unramified, and $\epsilon = 1$ if E/F is ramified.

Proof. (*i*) Let L be a standard lattice representing v. Then gL is also a standard lattice representing gv. Then $g(1_v) = 1_{gv}$ and $g(\xi_L) = \xi_{gL}$ so

$$\iota(g(1_v))(g(\xi_L)) = 1$$

which shows that $\iota(g(1_v)) = \iota(1_v) \circ g^{-1} = g(\iota(1_v))$.

If $\xi \in \mathcal{F}$ we define the height of a vertex v (w.r.t. origin v_0 and ξ) as follows. Let $v_0, v_1, \ldots, v_r, \ldots$ be the geoedesic from v_0 to ξ (recall that $\mathcal{F}$ is identified with the ends of the tree). Let k be such that v_k lies on the geodesic from v to ξ. Define

$$h_\xi(v) = k - d(v, v_k)$$

This is independent of k. If we "hang" the tree down from ξ, vertices with the same height are "equidistant" from ξ.

Consider $\phi = 1_v$. If $v \in \mathcal{T}_0^0$ we have

$$T_1(1_v) = \sum_{d(u,v)=2} 1_u$$

One of the neighbouring vertices, say u_0, lies closer to ξ than v, i.e. $h_\xi(u_0) = h_\xi(v) + 1$, and the other $|N_0/N_1|$ vertices, say u_i $(1 \le i \le |N_0/N_1|)$ satisfy $h_\xi(u_i) = h_\xi(v) - 1$.

The vertex u_0 has a single neighbour, w_0, which lies even closer to ξ, so that $h_\xi(w_0) = h_\xi(v) + 2$, and the other $\left|\overline{N}_1/\overline{N}_2\right| - 1$ vertices, say w_i $\left(1 \le i \le \left|\overline{N}_1/\overline{N}_2\right| - 1\right)$ satisfy $h_\xi(w_i) = h_\xi(v)$.

Any of the u_i $(i > 0)$ has $\left|\overline{N}_1/\overline{N}_2\right|$ other neighbours, w_{ij} $\left(0 \le j < \left|\overline{N}_1/\overline{N}_2\right|\right)$ which lie further from ξ, hence $h_\xi(w_{ij}) = h_\xi(v) - 2$ for all $i > 0$ and all j.

But for any $\widehat{\xi}$ above ξ,

$$\iota(1_{w_0})(\hat{\xi}) = \iota(1_v)(\hat{\xi}) \cdot t^{-1}$$

while if $1 \leq i < \left|\overline{N}_1/\overline{N}_2\right|$, then

$$\iota(1_{w_i})(\hat{\xi}) = \iota(1_v)(\hat{\xi})$$

and if $i > 0$ and j is arbitrary, then

$$\iota(1_{w_{ij}})(\hat{\xi}) = \iota(1_v)(\hat{\xi}) \cdot t$$

In fact, the formula

$$\iota(1_v)(\xi_{L_0}) = t^{\left[\frac{h_\xi(v)}{2}\right]}$$

holds true, if $L_0 = \mathcal{O}_E^3$ is the standard lattice representing v_0.
It follows, using Lemma 2.23, that for $v \in \mathcal{T}_0^0$, if E/F is unramified

$$\iota\left(T_1(1_v)\right) = \iota(1_v) \cdot (t^{-1} + (q^{1/2} - 1) + t \cdot q^2)$$

while if E/F is ramified

$$\iota\left(T_1(1_v)\right) = \iota(1_v) \cdot (t^{-1} + (q - 1) + t \cdot q^2)$$

This proves (i)
(ii) Specializing t to λ, we obtain an intertwining operator $\iota_\lambda : ind_{K_0}^G C \to ind_B^G \chi_\lambda$ which factors through $(T_1 - \lambda^{-1} - \lambda \cdot q^2 - (q^\epsilon - 1)) \cdot ind_{K_0}^G C$, and sends 1_{K_0} to the spherical vector ϕ. As both ι_λ, j are G-equivariant, and 1_{K_0} generates $ind_{K_0}^G C$ as a G-representation, they are both determined by the image of 1_{K_0}. But $\iota_\lambda(1_{K_0}) = \phi = j(1_{K_0})$, hence $\iota_\lambda = j$.
As $ind_B^G \chi_\lambda$ is irreducible, it follows from Theorem 3.12 that $\iota_\lambda = j$ induces an isomorphism

$$\pi(\chi_\lambda) \simeq \frac{ind_{K_0}^G C}{(T_1 - \alpha_\chi(T_1)) \cdot ind_{K_0}^G C}$$

where T_1 acts as $\alpha_\chi(T_1)$ on $\pi(\chi_\lambda)$. But we have shown in (i), that T_1 acts on $\pi(\chi_\lambda)$ as $\lambda^{-1} + \lambda \cdot q^2 + (q^\epsilon - 1)$, thus we obtain an isomorphism

$$\pi(\chi_\lambda) \simeq \frac{ind_{K_0}^G C}{(T_1 - \lambda^{-1} - \lambda \cdot q^2 - (q^\epsilon - 1)) \cdot ind_{K_0}^G C}$$

□

Corollary 3.15. *Let $\chi : E^\times \to C^\times$ be an unramified character, such that $\chi(\pi) = \lambda$ and such that $ind_B^G \chi$ is an irreducible representation of G. Let $\underline{a}, \underline{b} \in \mathbb{Z}_{\geq 0}^{Hom_{alg}(E,C)}$, $\underline{d} \in \mathbb{Z}^{Hom_{alg}(E,C)}$, and denote $\pi^{\underline{a},\underline{b},\underline{d}} = \prod_{\tau : E \hookrightarrow C} \tau(\overline{\pi})^{a_\tau + d_\tau} \cdot \tau(\pi)^{b_\tau - d_\tau} \subseteq C$. Then*

$$ind_B^G \chi \otimes \rho_{\underline{a},\underline{b},\underline{d}} \simeq \frac{ind_{K_0}^G \rho_{\underline{a},\underline{b},\underline{d}}}{(T - \pi^{\underline{a},\underline{b},\underline{d}} \cdot (\lambda^{-1} - \lambda \cdot q^2 - (q^\epsilon - 1))) \cdot ind_{K_0}^G \rho_{\underline{a},\underline{b},\underline{d}}}$$

with ϵ as above.

We denote for any $c \in C$

$$\Pi_{\underline{a},\underline{b},c} = \frac{ind_{K_0}^G \rho_{\underline{a},\underline{b},\underline{0}}}{(T-c)\cdot ind_{K_0}^G \rho_{\underline{a},\underline{b},\underline{0}}}$$

where $\underline{0} : Hom_{alg}(E,C) \to \mathbb{Z}$ is the zero map, and will later translate using the above corollary conditions on c to conditions on λ.
We shall denote from now on also

$$\rho_{\underline{a},\underline{b}} := \rho_{\underline{a},\underline{b},\underline{0}}, \quad \rho(a_\tau, b_\tau)^\tau := \rho(a_\tau, b_\tau, 0)^\tau, \quad \pi^{\underline{a},\underline{b}} := \pi^{\underline{a},\underline{b},\underline{0}}$$

3.6. Integrality and separated lattices

Let us first define what does it mean for a representation of G to be integral.

Definition 3.16. Let R be a complete discrete valuation ring of fraction field S. An S-representation V of G of countable dimension with a basis generating a G-stable R-submodule of L, is called *integral* of *R-integral structure* L.

One may equivalently define integral structures as separated lattices in the following sense:

Definition 3.17. Let R be a complete discrete valuation ring of fraction field S. Let V be an S-representation of G. A *lattice* L in V is a sub-R-module of V such that for all $v \in V$, there exists a nonzero element $a \in S^\times$ such that $av \in L$. A lattice L is called *separated* if it does not contain any S-line, which is equivalent to $\bigcap_{n\in\mathbb{N}} \pi^n L = 0$, where $\pi \in R$ is a uniformizer.

The identification follows immediately:

Proposition 3.18. *Let R be a complete discrete valuation ring of fraction field S. Let V be an S-representation of G. An integral structure in V is a G-stable separated lattice in V.*

Example 3.19. The sub-$\mathcal{O}_C$-module $\rho^0(a_\tau, b_\tau)^\tau \subseteq \rho(a_\tau, b_\tau)^\tau$ for any $\tau : E \hookrightarrow C$ is defined by

$$\rho^0(a_\tau, b_\tau)^\tau = \left\{ f(x_\tau, y_\tau) \in \bigoplus_{\substack{i,j \in \mathbb{Z}^3_{\geq 0} \\ |i|=a_\tau, |j|=b_\tau}} \mathcal{O}_C \cdot x_\tau^i y_\tau^j \mid \sum_{k=1}^3 \frac{\partial^2 f}{\partial x_{\tau,k} \partial y_{\tau,k}} = 0 \right\}$$

It is a separated lattice in $\rho(a_\tau, b_\tau)^\tau$, which is further stable under the action of K_0.

Further, we note that U_{a_τ,b_τ} , defined in 3.1 acts as $\tau(\overline{\pi})^{a_\tau} \cdot \tau(\pi)^{b_\tau} \cdot \alpha^{-1}$, hence preserves $\rho^0(a_\tau, b_\tau)^\tau$. Moreover, it satisfies

$$U_{a_\tau,b_\tau}(x_{\tau,3}^{a_\tau} y_{\tau,1}^{b_\tau}) = x_{\tau,3}^{a_\tau} y_{\tau,1}^{b_\tau}$$

and for any other such vector (i.e. $(i,j) \neq (0,0,a_\tau),(b_\tau,0,0)$), we see that $x_\tau^i y_\tau^j \in \pi \cdot \rho^0(a_\tau, b_\tau)^\tau$.

Example 3.20. Therefore, we may also consider the sub-$\mathcal{O}_C$-module $\rho^0_{\underline{a},\underline{b}} = \bigotimes_{\tau:E\hookrightarrow C} \rho^0(a_\tau, b_\tau)^\tau \subset \rho_{\underline{a},\underline{b}}$ for any $\underline{a}, \underline{b} \in (\mathbb{Z}_{\geq 0})^{Hom_{alg}(E,C)}$. Then $\rho^0_{\underline{a},\underline{b}}$ is a separated lattice in $\rho_{\underline{a},\underline{b}}$ which is stable under the action of K_0.
Consequently, the $\mathcal{O}_C$-module $ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$ is also a separated lattice in $ind_{K_0}^G \rho_{\underline{a},\underline{b}}$, which is further stable under the action of G.

It follows that we have an injective map $\mathcal{H}_{\rho^0_{\underline{a},\underline{b}}}(K_0, G) \hookrightarrow \mathcal{H}_{\rho_{\underline{a},\underline{b}}}(K_0, G)$, and the operator $T \in \mathcal{H}_{\rho_{\underline{a},\underline{b}}}(K_0, G)$, defined in Lemma 3.9, induces by restriction a G-equivariant endomorphism of $\mathcal{H}_{\rho^0_{\underline{a},\underline{b}}}(K_0, G)$, which we denote by T as well.

Lemma 3.21. *There is an isomorphism of $\mathcal{O}_C$-algebras $\mathcal{H}_{\rho^0_{\underline{a},\underline{b}}}(K_0, G) \simeq \mathcal{O}_C[T]$.*

Proof. Recall that by Lemma 3.11, we have an isomorphism of C-algebras $\mathcal{H}_{\rho_{\underline{a},\underline{b}}}(K_0, G) \simeq C[T]$.
Moreover, the equations describing T_n as polynomial in T_1, show that $T_n \in \mathcal{O}_C[T_1]$, for all n. In fact, as T_n can be expressed as a monic polynomial of degree n in T_1 with integral coefficients, it follows that $(\pi^{\underline{a},\underline{b}})^n T_n$ can be expressed as a monic polynomial of degree n in $\pi^{\underline{a},\underline{b}} T_1 = T$, so that $(\pi^{\underline{a},\underline{b}})^n T_n \in \mathcal{O}_C[T]$. Let us write $f(T) = (\pi^{\underline{a},\underline{b}})^n T_n$ for this polynomial.
Let us show that its image on $\mathcal{H}_{\rho^0_{\underline{a},\underline{b}}}(K_0, G)$ is exactly $\mathcal{O}_C[T]$.
As $T \in \mathcal{H}_{\rho^0_{\underline{a},\underline{b}}}(K_0, G)$, clearly $\mathcal{O}_C[T]$ is contained in the image. Let $p(T) \in C[T]$ be a polynomial corresponding to an element in $\mathcal{H}_{\rho^0_{\underline{a},\underline{b}}}(K_0, G)$.
Assume $\deg(p) = n$, and let a_n be the leading coefficient, i.e. $p(T) \equiv a_n T^n + p_{n-1}(T)$, where $\deg(p_{n-1}) = n-1$. It follows that $p(T) \equiv a_n(\pi^{\underline{a},\underline{b}})^n T_n + q_{n-1}(T)$, for some q with $\deg(q_{n-1}) = n - 1$.
We recall that T_n is the image under the natural isomorphisms of $\mathbf{1}_{K_0\alpha^{-n}K_0} \in H_K(C)$, which maps to $\mathbf{1}_{K_0\alpha^{-n}K_0} \cdot \rho_{\underline{a},\underline{b}} \in H_K(\rho)$, finally mapping to

$$T_n([g, v]) = \sum_{xK_0 \in G/K_0} [gx, \mathbf{1}_{K_0\alpha^{-n}K_0}(x^{-1}) \rho_{\underline{a},\underline{b}}(x^{-1})(v)] =$$

$$= \sum_{xK_0 \in K_0\alpha^{-n}K_0/K_0} [gx, \rho_{\underline{a},\underline{b}}(x^{-1})(v)]$$

Since $\alpha^n \in K_0\alpha^{-n}K_0$, and polynomials of order less than n are supported on $\coprod_{i=0}^{n-1} K_0\alpha^{-i}K_0$, it follows that for any $v \in \rho_{\underline{a},\underline{b}}$, one has

$$(p(T)([1, v]))(\alpha^n) = \left(a_n(\pi^{\underline{a},\underline{b}})^n T_n([1, v])\right)(\alpha^n) = a_n(\pi^{\underline{a},\underline{b}})^n \rho_{\underline{a},\underline{b}}(\alpha^{-n})(v) = a_n U^n_{\underline{a},\underline{b}}(v)$$

where the right most equality follows from (3.2).
In particular, taking $v = \bigotimes_{\tau:E\hookrightarrow C} x_{\tau,3}^{a_\tau} y_{\tau.1}^{b_\tau}$, we see that $v \in \rho^0_{\underline{a},\underline{b}}$, hence $[1, v] \in ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$. As we assume $p(T) \in \mathcal{H}_{\rho^0_{\underline{a},\underline{b}}}(K_0, G) = End_{\mathcal{O}_C[G]}(ind_{K_0}^G \rho^0_{\underline{a},\underline{b}})$, it follows that $p(T)([1, v]) \in ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$, hence $a_n U^n_{\underline{a},\underline{b}}(v) = (p(T)([1, v]))(\alpha^n) \in \rho^0_{\underline{a},\underline{b}}$. But, by definition of U, we see that $U_{\underline{a},\underline{b}}(v) = v$, hence $a_n v \in \rho^0_{\underline{a},\underline{b}}$.

However, by definition of $\rho^0_{\underline{a},\underline{b}}$, this is possible if and only if $a_n \in \mathcal{O}_C$. Therefore, we see that $a_n T^n \in \mathcal{O}_C[T]$, and it suffices to prove the claim for $p(T) - a_n T^n$, which is a polynomial of degree less than n.

Proceeding by induction, where the induction basis consists of constant polynomials, who can be integral if and only if they belong to $\mathcal{O}_C$, we conclude that $p(T) \in \mathcal{O}_C[T]$. □

4. Integrality in unramified principal series representations

As we are interested in studying the integrality of irreducible locally algebraic representations, we consider representations of the form $ind_B^G(\chi \otimes \chi_1) \otimes \rho_{\underline{a},\underline{b},\underline{d}}$.

When discussing the existence of an integral structure, we may twist by a central character, hence one may assume that $\chi_1 = 1$.

In addition, for any $g \in G$, $\det(g) \in \mathcal{O}_E^\times$ is a unit, hence integrality is not affected by twists of the determinant, and we may assume that $\underline{d} = \underline{0}$.

Moreover, by Corollary 3.15, we may consider only representations of the form $\Pi_{\underline{a},\underline{b},c}$.

4.1. Construction of Lattice

In order to hope for an integral structure in $\Pi_{\underline{a},\underline{b},c}$, we should demand that $c \in \mathcal{O}_C$. We intend to show that this is a sufficient condition, at least when $\underline{a}, \underline{b}$ are small.

We may now define for any $\alpha \in \mathcal{O}_C$

$$\Theta_{\underline{a},\underline{b},c} = \mathrm{Im}\left(ind_{K_0}^G \rho^0_{\underline{a},\underline{b}} \to \Pi_{\underline{a},\underline{b},c}\right)$$

This is a lattice in $\Pi_{\underline{a},\underline{b},c}$, and as $ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$ is an $\mathcal{O}_C[G]$-module of finite type, we see that $\Theta_{\underline{a},\underline{b},c}$ is also an $\mathcal{O}_C[G]$-module of finite type.

Conjecture 4.1. *If $c \in \mathcal{O}_C$, then $\Theta_{\underline{a},\underline{b},c}$ is an integral structure in $\Pi_{\underline{a},\underline{b},c}$.*

Let $\epsilon = \frac{1}{2}$ if E/F is unramified, and $\epsilon = 1$ if E/F is ramified.

Note that by Corollary 3.15, as $q^\epsilon - 1$ is a unit, this implies

$$|\pi|^{a+b} \cdot \max\left(\left|\lambda^{-1}\right|, \left|\lambda \cdot q^2\right|, 1\right) = \left|\pi^{\underline{a},\underline{b}}\right| \cdot \left|\lambda^{-1} + \lambda \cdot q^2 + q^\epsilon - 1\right| \le 1$$

where the equality on the left holds since $|\lambda^{-1}| = |\lambda \cdot q^2| \Rightarrow |\lambda| = |q|^{-1} > 1$, whence $|\lambda^{-1}| = |q| < 1$.

where $a = \sum_{\tau: E \hookrightarrow C} a_\tau$, $b = \sum_{\tau: E \hookrightarrow C} b_\tau$. We obtain

$$\pi^{a+b} \cdot \chi(\pi)^{-1} \in \mathcal{O}_C, \quad \pi^{a+b} \cdot q^2 \cdot \chi(\pi) \in \mathcal{O}_C$$

which is the condition stated in Theorem 1.5.

We will further assume that $c \in \mathfrak{p}_C$. The results will hold also trivially for $c \in \mathcal{O}_C^\times$.

As T stabilizes $ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$, we have

$$(T-c)(ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}) \subseteq (T-c)(ind_{K_0}^G \rho_{\underline{a},\underline{b}}) \cap ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$$

Thus we have a surjective homomorphism of $\mathcal{O}_C[G]$-modules

$$\theta : \frac{ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}}{(T-c) ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}} \twoheadrightarrow \Theta_{\underline{a},\underline{b},c}$$

When θ is injective, $\Theta_{\underline{a},\underline{b},c}$ is a quotient of a free $\mathcal{O}_C$-module. However, such a quotient might contain a C-line.

As the following simple Lemma suggests, the missing ingredient is the completeness of the quotient ideal, which is clear in the above case.

Lemma 4.2. *Let M be an $\mathcal{O}_C$-module. Let $N \subset M$ be a submodule such that $N \otimes_{\mathcal{O}_C} C$ is π_C-adically complete, and such that $(N \otimes_{\mathcal{O}_C} C) \cap M = N$. Then M/N contains no C-line.*

Proof. Assume that $C \cdot [m] \subset M/N$ for some $m \in M$. Then for any $n \in \mathbb{N}$, there exists $m_n \in M$ such that $[m_n] = \pi_C^{-n}[m]$, i.e. $m - \pi_C^n m_n \in N$.

But π_C-adically, we see that $\lim_{n\to\infty}(m - \pi_C^n m_n) = m$, hence by completeness, $m \in N \otimes_{\mathcal{O}_C} C$. Thus, $m \in (N \otimes_{\mathcal{O}_C} C) \cap M = N$, so that $[m] = 0$. This shows that M/N contains no C-line. $\square$

Consequently, if θ is injective, letting $M = ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$ and $N = (T-c) \cdot M$, we see that they satisfy the assumptions of the Lemma, establishing that $\Theta_{\underline{a},\underline{b},c}$ contains no C-line, hence the conjecture is true. Therefore, we are interested in determining when is θ injective.

The following theorem is inspired by the work (Große-Klönne [14]), who proves a generalization of this statement for split reductive groups, and uses the ideas presented there.

Theorem 4.3. *The homorphism θ is injective iff $\left(\rho^0_{\underline{a},\underline{b}} \otimes k_C\right)^{I(1)}$ is one-dimensional.*

Proof. Begin by showing the "if " statement. Let $f \in ind_{K_0}^G \rho_{\underline{a},\underline{b}}$ be such that $T(f) - cf \in ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$.

Recall that for any $n \geq 0$ we have defined $S_n = K_0 \alpha^{-n} K_0$, and $B_n = \coprod_{m=0}^n S_m$. Moreover, we have $S_m = S_m^0 \coprod S_m^1$, where $S_m^0 = I\alpha^{-m}K_0$, $S_m^1 = I\beta\alpha^{1-m}K_0$.

We further denote, for any $\mathcal{O}_C$-module V, by $B_n(V), S_n(V), S_n^i(V)$ the functions supported on B_n, S_n, S_n^i respectively, with values in V.

Let n be the minimal integer such that $f \in B_n(\rho_{\underline{a},\underline{b}})$. Write $f = \sum_{m=0}^n f_m$, with $f_m \in S_m(\rho_{\underline{a},\underline{b}})$. For any m, we let $f_m = f_m^0 + f_m^1$ with $f_m^i \in S_m^i(\rho_{\underline{a},\underline{b}})$.

Recall that by our earlier observations (Corollary 2.19)

$$K_0\alpha^{-1}K_0 = I\alpha^{-1}K_0 \coprod I\beta K_0 = \left(\coprod_{\eta\in N_0/N_2} \eta\alpha^{-1}K_0\right) \coprod \left(\coprod_{\eta\in \overline{N}_1/\overline{N}_2} \eta\beta K_0\right)$$

Let$\psi \in \mathcal{H}_{K_0}(\rho_{\underline{a},\underline{b}})$ be the function corresponding to T, as defined in Lemma 3.1. It now follows from (3.4) that for any $g \in G$ and any $v \in \rho_{\underline{a},\underline{b}}$

$$T([g,v]) = \sum_{\eta\in N_0/N_2} [g\eta\alpha^{-1}, \psi(\alpha)\circ\eta^{-1}(v)] + \sum_{\eta\in\overline{N}_1/\overline{N}_2} [g\eta\beta, \psi(\beta^{-1})\circ\eta^{-1}(v)]$$

where we have used the fact that ψ is K_0-bi-equivariant.
Let us denote

$$T^+([g,v]) = \sum_{\eta\in N_0/N_2} [g\eta\alpha^{-1}, \psi(\alpha)\circ\eta^{-1}(v)]$$

$$T^0([g,v]) = \sum_{1\neq\eta\in\overline{N}_1/\overline{N}_2} [g\eta\beta, \psi(\beta^{-1})\circ\eta^{-1}(v)]$$

and

$$T^-([g,v]) = [g\beta, \psi(\beta^{-1})(v)]$$

Then $T^+(f_n^0)$ is supported on S_{n+1}, and as cf is supported on B_n, it follows by assumption that $T^+(f_n^0) \in ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$.

By the following Lemma 4.4, it will follow that $f_n^0 \in ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$.

Similarly, since $f_n^1 \in S_n^1$, we see that $\beta f_n^1 \in S_{n-1}^0$, hence $T^+(\beta f_n^1) \in S_n^0$ and $\beta T^+(\beta f_n^1)$ is supported on S_{n+1}. Since cf is supported on B_n, by assumption, we see that also $\beta T^+(\beta f_n^1) \in ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$.

However, it follows that $T^+(\beta f_n^1) \in ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$, hence by the Lemma 4.4, we also have $\beta f_n^1 \in ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$, showing that $f_n^1 \in ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$. Thus, $f_n = f_n^0 + f_n^1 \in ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$.

Proceeding by induction, we see that $f \in ind_{K_0}^G \rho^0_{\underline{a},\underline{b}}$, hence the result.

Conversely, if $\left(\rho^0_{\underline{a},\underline{b}} \otimes k_C\right)^{I(1)}$ is not one dimensional, it must contains $v \neq \bigotimes_\tau x_{\tau,1}^{a_\tau} y_{\tau,3}^{b_\tau}$. Moreover, as $\bigotimes_\tau x_{\tau,3}^{a_\tau} y_{\tau,1}^{b_\tau}$ is not $I(1)$-invariant (unless the representation is trivial, hence one-dimensional), it follows that $sv \neq \bigotimes_\tau x_{\tau,1}^{a_\tau} y_{\tau,3}^{b_\tau}$. Therefore $\psi(\alpha)$ acts as a multiple of π, by construction, on both v and sv.

By rescaling, we may further assume that $v \notin \pi\rho^0_{\underline{a},\underline{b}}$. Let $\delta \in \{\iota(\pi), c\}$ be of minimal valuation. (Here $\iota : E \hookrightarrow C$ is a choice of a fixed embedding) It follows that $\delta^{-1}\iota(\pi), \delta^{-1}c \in \mathcal{O}_C$, hence

$$T^+([1,\delta^{-1}v]) = \sum_{\eta\in N_0/N_2} [\eta\alpha^{-1}, \psi(\alpha)\circ\eta(\delta^{-1}v)] =$$

$$= \sum_{\eta \in N_0/N_2} [\eta\alpha^{-1}, \delta^{-1}\psi(\alpha)(v)] \in ind_{K_0}^{G}\rho^0_{\underline{a},\underline{b}}$$

also, as for any $\eta \in \overline{N}_1$, $\eta \in 1 + \pi K_0$, we see that

$$T^0([1,\delta^{-1}v]) + T^-([1,\delta^{-1}v]) = \sum_{\eta \in \overline{N}_1/\overline{N}_2} [\eta\beta, \psi(\beta^{-1}) \circ \eta^{-1}(\delta^{-1}v)] =$$

$$= \sum_{\eta \in \overline{N}_1/\overline{N}_2} [\eta\beta, \delta^{-1}\psi(\pm\alpha s) \circ \eta^{-1}(v)] \in \sum_{\eta \in \overline{N}_1/\overline{N}_2} [\eta\beta, \delta^{-1}\psi(\pm\alpha)(sv)] + ind_{K_0}^{G}\rho^0_{\underline{a},\underline{b}}$$

where the sign is determined by ramification of E/F. As $\psi(\alpha)$ acts as a multiple of π also on sv, we deduce that $T([1,\delta^{-1}v]) \in ind_{K_0}^{G}\rho^0_{\underline{a},\underline{b}}$. However, also $c \cdot [1,\delta^{-1}v] = [1,(\delta^{-1}c)v] \in ind_{K_0}^{G}\rho^0_{\underline{a},\underline{b}}$, showing that

$$(T-c)([1,\delta^{-1}v]) \in ind_{K_0}^{G}\rho^0_{\underline{a},\underline{b}} \cap (T-c)ind_{K_0}^{G}\rho_{\underline{a},\underline{b}}$$

while $[1,\delta^{-1}v] \notin ind_{K_0}^{G}\rho^0_{\underline{a},\underline{b}}$. Thus θ is not injective. □

Lemma 4.4. *Let $n \in \mathbb{N}$, and let $f_n^0 \in S_n^0(\rho_{\underline{a},\underline{b}})$ be such that $T^+(f_n^0) \in ind_{K_0}^{G}\rho^0_{\underline{a},\underline{b}}$. Assume $\left(\rho^0_{\underline{a},\underline{b}} \otimes k_C\right)^{I(1)}$ is one-dimensional. Then $f_n^0 \in ind_{K_0}^{G}\rho^0_{\underline{a},\underline{b}}$.*

Proof. It suffices to show it for a function of the form $[g,v]$, but then by the above formula

$$T^+([g,v]) \in ind_{K_0}^{G}\rho^0_{\underline{a},\underline{b}} \iff \psi(\alpha) \circ \eta^{-1}(v) \in \rho^0_{\underline{a},\underline{b}} \quad \forall \eta \in N_0/N_2$$

So we would like to show that this holds iff $v \in \rho^0_{\underline{a},\underline{b}}$. We note further that

$$v \mapsto \big((\psi(\alpha) \circ \eta^{-1})(v)\big)_{\eta \in N_0/N_2} : \rho_{\underline{a},\underline{b}} \to (\rho_{\underline{a},\underline{b}})^{|N_0/N_2|}$$

is a linear map, and moreover, the matrix representing it has integral coefficients. Hence it is enough to show that its reduction mod π_C is injective, which will show that the map on the lattice is invertible, hence our result.

Note that $N_2 = \alpha^{-1}N_0\alpha$. Therefore, we have a natural conjugation map $n_2 \mapsto \alpha n_2 \alpha^{-1} : N_2 \to N_0$, so that any N_0-module is also an N_2-module.

We use it to define for any N_0-module V

$$ind_{N_2}^{N_0}V = \left\{ f : N_0 \to V \mid f(n_0 n_2) = \alpha n_2 \alpha^{-1} f(n_2) \quad \forall n_0 \in N_0, n_2 \in N_2 \right\}$$

with N_0 acting by left translation.

Consider the map

$$\begin{array}{rcl} \rho^0_{\underline{a},\underline{b}} \otimes k_C & \to & ind_{N_2}^{N_0}\rho^0_{\underline{a},\underline{b}} \otimes k_C \\ v & \mapsto & [n \mapsto \psi(\alpha)(n^{-1}v)] \end{array}$$

It is a map of N_0-modules, which we want to show is injective.

Next, we note that N_0 is pro-p, and let U be the kernel of this map. If $U \neq 0$, then $U^{N_0} \neq 0$, hence it suffices to show that the map

$$\begin{array}{rcl}(\rho^0_{\underline{a},\underline{b}} \otimes k_C)^{N_0} & \to & \left(ind_{N_2}^{N_0} \rho^0_{\underline{a},\underline{b}} \otimes k_C\right)^{N_0} \simeq (\rho^0_{\underline{a},\underline{b}} \otimes k_C)^{N_2} \\ v & \mapsto & [n \mapsto \psi(\alpha)(n^{-1}v)] \mapsto \psi(\alpha)(v)\end{array}$$

is injective.

But $(\rho^0_{\underline{a},\underline{b}} \otimes k_C)^{N_0} = (\rho^0_{\underline{a},\underline{b}} \otimes k_C)^{I(1)}$ is one-dimensional, and clearly contains the vector $\bigotimes_\tau x_{\tau,1}^{a_\tau} y_{\tau,3}^{b_\tau}$, on which $\psi(\alpha) = \pm s \circ \psi(\alpha^{-1}) \circ s$ acts as ± 1, depending on the ramification of E/F, by definition. Therefore, the above map is simply the identity. $\square$

We may now make this criterion more explicit in terms of a_τ, b_τ .

First, we note that if the weights are all p-restricted, then the reduction $\rho^0_{\underline{a},\underline{b}} \otimes k_C$ remains irreducible (see, e.g. Jantzen [16]). In particular, $\left(\rho^0_{\underline{a},\underline{b}} \otimes k_C\right)^{I(1)}$ is the line spanned by the highest weight vector, hence one-dimensional, and we obtain

Corollary 4.5. *If for all $\tau : E \hookrightarrow C$, $0 \leq a_\tau, b_\tau < p$, then θ is injective, and $\Pi_{\underline{a},\underline{b},c}$ is integral for all $c \in \mathcal{O}_C$.*

Next, we consider several cases where we know θ fails to be injective.

Let e be the ramification index of E over $\mathbb{Q}_p$, and denote $q = p^f$, so that $[E : \mathbb{Q}_p] = ef$. Fix an embedding $\iota : E \hookrightarrow C$.

Denote by $S^+ = \{\tau : E \hookrightarrow C \mid a_\tau + b_\tau \neq 0\}$ the embeddings for which we have a nontrivial component in our representation.

For any $l \in \mathbb{Z}/f\mathbb{Z}$, let

$$J_l = \{\tau \in S^+ \mid \tau([\zeta]) = \iota([\zeta]^{p^l}) \quad \forall \zeta \in k_E\}$$

where $[\zeta]$ is the Teichmüller lift of ζ in E.

For any $\tau \in J_l$, we let

$$v_\tau = \inf \{1 \leq i \leq f \mid J_{l+i \mod f} \neq \emptyset\}$$

Note that v_τ does not depend on τ but only on the unique l such that $\tau \in J_l$.

Lemma 4.6. *Assume there exists $l \in \mathbb{Z}/f\mathbb{Z}$ such that $|J_l| > 1$. Then θ is not injective.*

Proof. By assumption, there exist distinct $\tau, \xi \in S^+$ such that $\xi([\zeta]) = \tau([\zeta])$ for all $\zeta \in k_E$. In this proof we shall denote by ρ the embeddings $\rho : E \hookrightarrow C$.

Assume first that both a_ξ, a_τ are nonzero. Then, setting $\underline{i}^\tau = (i_\rho^\tau), \underline{i}^\xi = (i_\rho^\xi), \underline{i} = (i_\rho), \underline{j} = (j_\rho)$ with

$$i_\rho^\tau = \begin{cases} (a_\rho - 1, 1, 0) & \rho = \tau \\ (a_\rho, 0, 0) & \rho \neq \tau \end{cases}, \quad i_\rho^\xi = \begin{cases} (a_\rho - 1, 1, 0) & \rho = \xi \\ (a_\rho, 0, 0) & \rho \neq \xi \end{cases},$$

$$i_\rho = (a_\rho, 0, 0), \quad j_\rho = (0, 0, b_\rho)$$

we may define $v = x^{\underline{i}^\tau} y^{\underline{j}} - x^{\underline{i}^\xi} y^{\underline{j}} \in \rho^0_{\underline{a},\underline{b}} \otimes k_C$. Let us show that v is $I(1)$-invariant. Indeed, $I(1)$ acts via the reduction, hence it suffices to consider $n_{\beta,\zeta} \in N_0$, with $\beta, \zeta \in \mathcal{O}_E$.
However,

$$n_{\beta,\zeta} \cdot x^{\underline{i}^\tau} y^{\underline{j}} = \bigotimes_{\rho : E \hookrightarrow C} ({}^{\mathrm{t}}n_{\rho(\beta),\rho(\zeta)} \cdot x_\rho)^{i_\rho^\tau} \cdot (n_{\rho(-\beta),\rho(\overline{\zeta})} \cdot y_\rho)^{j_\rho} =$$

$$= \left(\bigotimes_{\rho \neq \tau} x_{\rho,1}^{a_\rho} \cdot y_{\rho,3}^{b_\rho} \right) \otimes x_{\tau,1}^{a_\tau - 1} \cdot (x_{\tau,2} + \tau(\beta) x_{\tau,1}) \cdot y_{\tau,3}^{b_\tau} = x^{\underline{i}^\tau} y^{\underline{j}} + \tau(\beta) \cdot x^{\underline{i}} y^{\underline{j}}$$

and similarly, replacing τ by ξ, we get

$$n_{\beta,\zeta} \cdot x^{\underline{i}^\xi} y^{\underline{j}} = x^{\underline{i}^\xi} y^{\underline{j}} + \xi(\beta) \cdot x^{\underline{i}} y^{\underline{j}} \tag{4.1}$$

Therefore

$$n_{\beta,\zeta} \cdot v = v + (\tau(\beta) - \xi(\beta)) \cdot x^{\underline{i}} y^{\underline{j}}$$

However, as $\rho(\pi_E) \in \pi_C \mathcal{O}_C$ for all $\rho : E \hookrightarrow C$, in particular for τ, ξ, we see that in k_C, $\tau(\beta) - \xi(\beta)$ depends only on $\beta \mod \pi_E$. Since it vanishes on the Teichmuller lifts, it vanishes everywhere, hence $n_{\beta,\zeta} \cdot v = v$, showing that v is $I(1)$-invariant, hence by Theorem 4.3, θ is not injective.
Similarly, if both b_ξ, b_τ are nonzero, we may consider $\underline{j}^\tau, \underline{j}^\xi$ defined by

$$j_\rho^\tau = \begin{cases} (0, 1, b_\rho - 1) & \rho = \tau \\ (0, 0, b_\rho) & \rho \neq \tau \end{cases}, \quad j_\rho^\xi = \begin{cases} (0, 1, b_\rho - 1) & \rho = \xi \\ (0, 0, b_\rho) & \rho \neq \tau \end{cases}$$

and $v = x^{\underline{i}} y^{\underline{j}^\tau} - x^{\underline{i}} y^{\underline{j}^\xi} \in \rho^0_{\underline{a},\underline{b}} \otimes k_C$, then v is $I(1)$-invariant. Indeed

$$n_{\beta,\zeta} \cdot x^{\underline{i}} y^{\underline{j}^\tau} = x^{\underline{i}} y^{\underline{j}^\tau} + \tau(\overline{\beta}) \cdot x^{\underline{i}} y^{\underline{j}}$$

and $\tau(\overline{\beta}) = \xi(\overline{\beta})$ in k_C, showing that $n_{\beta,\zeta} \cdot v = v$. Hence, again, by Theorem 4.3, θ is not injective.
Finally, if w.l.o.g. $b_\tau = a_\xi = 0$, let us denote by $\sigma : E \to E$ the conjugation. Then $\xi \circ \sigma : E \hookrightarrow C$ is also an embedding. We may consider $v = x^{\underline{i}^\tau} y^{\underline{j}} - x^{\underline{i}} y^{\underline{j}^{\xi \circ \sigma}}$. Then v is $I(1)$-invariant. Indeed

$$n_{\beta,\zeta} \cdot v = x^{\underline{i}^\tau} y^{\underline{j}} + \tau(\beta) \cdot x^{\underline{i}} y^{\underline{j}} - \left(x^{\underline{i}} y^{\underline{j}^{\xi \circ \sigma}} + \xi \circ \sigma(\overline{\beta}) \cdot x^{\underline{i}} y^{\underline{j}} \right) = v + (\tau(\beta) - \xi(\beta)) \cdot x^{\underline{i}} y^{\underline{j}} = v$$

and once more, by Theorem 4.3, θ is not injective. □

Lemma 4.7. *If there exists $\rho \in J_l$ such that either $a_\rho \geq p^{v_\rho}$ or $b_\rho \geq p^{v_\rho}$, then θ is not injective.*

Proof. We distinguish three possible cases:
(i) $|S^+| \geq 2$.
(ii) $|S^+| = 1$, and either $a_\rho \geq p^{v_\rho}$ or $b_\rho \geq p^{v_\rho}$.
<u>Case (i) :</u>
We may assume that $v_\rho < f$, else $S^+ = J_l$, so that $|J_l| \geq 2$, and we are done by Lemma 4.6.
Therefore, there exists $\tau \in S^+$ such that for all $\zeta \in k_E$, $\rho([\zeta])^{p^{v_\rho}} = \tau([\zeta])$. Assume first that $a_\rho \geq p^{v_\rho}$. Consider $\underline{i}^\rho$, $\underline{i}^\tau$, $\underline{i}, \underline{j}$ defined by

$$i_\xi^\rho = \begin{cases} (a_\rho - p^{v_\rho}, p^{v_\rho}, 0) & \rho = \xi \\ (a_\xi, 0, 0) & \rho \neq \xi \end{cases}, \quad i_\xi^\tau = \begin{cases} (a_\tau - 1, 1, 0) & \tau = \xi \\ (a_\xi, 0, 0) & \tau \neq \xi \end{cases},$$

$$i_\xi = (a_\xi, 0, 0), \quad j_\xi = (0, 0, b_\xi)$$

and let $v = x^{\underline{i}^\rho} y^{\underline{j}} - x^{\underline{i}^\tau} y^{\underline{j}} \in \rho^0_{\underline{a},\underline{b}} \otimes k_C$. Then v is $I(1)$-invariant. Indeed, for $\beta, \zeta \in \mathcal{O}_E$

$$n_{\beta,\zeta} \cdot x^{\underline{i}^\rho} y^{\underline{j}} = \left(\bigotimes_{\rho \neq \xi} x_{\xi,1}^{a_\xi} \cdot y_{\xi,3}^{b_\xi} \right) \otimes x_{\rho,1}^{a_\rho - p^{v_\rho}} \cdot (x_{\rho,2} + \rho(\beta) x_{\rho,1})^{p^{v_\rho}} \cdot y_{\rho,3}^{b_\rho} =$$

$$= \left(\bigotimes_{\rho \neq \xi} x_{\xi,1}^{a_\xi} \cdot y_{\xi,3}^{b_\xi} \right) \otimes x_{\rho,1}^{a_\rho - p^{v_\rho}} \cdot \left(x_{\rho,2}^{p^{v_\rho}} + \rho(\beta)^{p^{v_\rho}} x_{\rho,1}^{p^{v_\rho}} \right) \cdot y_{\rho,3}^{b_\rho} = x^{\underline{i}^\rho} y^{\underline{j}} + \rho(\beta)^{p^{v_\rho}} \cdot x^{\underline{i}} y^{\underline{j}}$$

and in (4.1) we have seen that

$$n_{\beta,\zeta} \cdot x^{\underline{i}^\tau} y^{\underline{j}} = x^{\underline{i}^\tau} y^{\underline{j}} + \tau(\beta) \cdot x^{\underline{i}} y^{\underline{j}}$$

Therefore

$$n_{\beta,\zeta} \cdot v = v + \left(\rho(\beta)^{p^{v_\rho}} - \tau(\beta) \right) \cdot x^{\underline{i}} y^{\underline{j}}$$

By the choice of ρ, τ, we have $\rho(\beta)^{p^{v_\rho}} = \tau(\beta)$ in k_C, thus showing that v is $I(1)$-invariant, and by Theorem 4.3, that θ is not injective.
Next, if $b_\rho \geq p^{v_\rho}$, we may consider $\underline{j}^\rho, \underline{j}^\tau$ defined by

$$j_\xi^\rho = \begin{cases} (0, p^{v_\rho}, b_\rho - p^{v_\rho}) & \rho = \xi \\ (0, 0, b_\xi) & \rho \neq \xi \end{cases}, \quad j_\xi^\tau = \begin{cases} (0, 1, b_\tau - 1) & \tau = \xi \\ (0, 0, b_\tau) & \tau \neq \xi \end{cases}$$

and let $v = x^{\underline{i}} y^{\underline{j}^\rho} - x^{\underline{i}} y^{\underline{j}^\tau}$. Then for $\beta, \zeta \in \mathcal{O}_E$

$$n_{\beta,\zeta} \cdot x^{\underline{i}} y^{\underline{j}^\rho} = \left(\bigotimes_{\rho \neq \xi} x_{\xi,1}^{a_\xi} \cdot y_{\xi,3}^{b_\xi} \right) \otimes x_{\rho,1}^{a_\rho} \cdot \left(y_{\rho,2} + \rho(\overline{\beta}) y_{\rho,3} \right)^{p^{v_\rho}} \cdot y_{\rho,3}^{b_\rho - p^{v_\rho}} =$$

$$= \left(\bigotimes_{\rho \neq \xi} x_{\xi,1}^{a_\xi} \cdot y_{\xi,3}^{b_\xi} \right) \otimes x_{\rho,1}^{a_\rho} \cdot \left(y_{\rho,2}^{p^{v_\rho}} + \rho(\overline{\beta})^{p^{v_\rho}} y_{\rho,3}^{p^{v_\rho}} \right) \cdot y_{\rho,3}^{b_\rho - p^{v_\rho}} = x^{\underline{i}} y^{\underline{j}^\rho} + \rho(\overline{\beta})^{p^{v_\rho}} \cdot x^{\underline{i}} y^{\underline{j}}$$

so that

$$n_{\beta,\zeta} \cdot v = v + \left(\rho(\overline{\beta})^{p^{v_\rho}} - \tau(\overline{\beta}) \right) \cdot x^{\underline{i}} y^{\underline{j}} = v$$

since the coefficients are in k_C. Therefore, v is $I(1)$-invariant, and by Theorem 4.3, θ is not injective.

Case (ii):

Since $|S^+| = 1$, $v_\rho = f$. Assume first that $a_\rho \geq p^{v_\rho} = p^f = q$. Set

$$v = x_\rho^{(a_\rho - 1, 1, 0)} y_\rho^{(0,0,b_\rho)} - x_\rho^{(a_\rho - q, q, 0)} y_\rho^{(0,0,b_\rho)}$$

Then for any $\beta, \zeta \in \mathcal{O}_E$ we see that

$$n_{\beta,\zeta} \cdot v = x_{\rho,1}^{a_\rho - 1} \cdot (x_{\rho,2} + \rho(\beta) \cdot x_{\rho,1}) \cdot y_{\rho,3}^{b_\rho} - x_{\rho,1}^{a_\rho - q} \cdot (x_{\rho,2} + \rho(\beta) \cdot x_{\rho,1})^q \cdot y_{\rho,3}^{b_\rho} =$$

$$= x_{\rho,1}^{a_\rho - 1} x_{\rho,2} y_{\rho,3}^{b_\rho} + \rho(\beta) \cdot x_{\rho,1}^{a_\rho} y_{\rho,3}^{b_\rho} - x_{\rho,1}^{a_\rho - q} x_{\rho,2}^q y_{\rho,3}^{b_\rho - q} - \rho(\beta)^q \cdot x_{\rho,1}^{a_\rho} y_{\rho,3}^{b_\rho} = v$$

where the last equality follows from the fact that in k_C, $\beta^q = \beta$. Thus v is $I(1)$-invariant, and by Theorem 4.3, θ is not injective.

If $b_\rho \geq p^{v_\rho} = p^f = q$, set

$$v = x_\rho^{(a_\rho,0,0)} y_\rho^{(0,1,b_\rho - 1)} - x_\rho^{(a_\rho,0,0)} y_\rho^{(0,q,b_\rho - q)}$$

Then for any $\beta, \zeta \in \mathcal{O}_E$ we see that

$$n_{\beta,\zeta} \cdot v = x_{\rho,1}^{a_\rho} \cdot (y_{\rho,2} + \rho(\overline{\beta}) \cdot y_{\rho,3}) \cdot y_{\rho,3}^{b_\rho - 1} - x_{\rho,1}^{a_\rho} \cdot (y_{\rho,2} + \rho(\overline{\beta}) \cdot y_{\rho,1})^q \cdot y_{\rho,3}^{b_\rho - q} =$$

$$= x_{\rho,1}^{a_\rho} y_{\rho,2} y_{\rho,3}^{b_\rho - 1} + \rho(\overline{\beta}) \cdot x_{\rho,1}^{a_\rho} y_{\rho,3}^{b_\rho} - x_{\rho,1}^{a_\rho} y_{\rho,2}^q y_{\rho,3}^{b_\rho - q} - \rho(\overline{\beta})^q \cdot x_{\rho,1}^{a_\rho} y_{\rho,3}^{b_\rho} = v$$

where the last equality follows from the fact that in k_C, $\overline{\beta}^q = \overline{\beta}$. Thus v is $I(1)$-invariant, and by Theorem 4.3, θ is not injective. □

This leads us to conjecture the following.

Conjecture 4.8. *θ is injective iff :*

1. For all $l \in \mathbb{Z}/f\mathbb{Z}$, $|J_l| \leq 1$.
2. For $\rho \in J_l$, one has $a_\rho, b_\rho < p^{v_\rho}$.

We have only proven that these conditions are necessary.

5. Diagrams, Coefficient Systems and induced representations

It has been brought to our attention that the material for this section can also be found in Koziol and Xu [19].

5.1. Coefficient systems

Coefficient systems were introduced over $\mathbb{C}$ by Schneider and Stuhler [21]. In this section, we follow Paskunas [20] and translate the language of coefficient systems to the group G.

The notation σ will be used throughout this section to denote a simplex (either a vertex or an edge in $\mathcal{T}$), as there is no risk for confusion with the previously defined embeddings $E \hookrightarrow C$.

Let $\mathcal{T}$ be the Bruhat-Tits tree of G. Let R be a commutative ring.

Definition 5.1. An *R-coefficient system* $\mathcal{V} = \{V_\sigma\}_\sigma$ on a simplicial set $\mathcal{T}$ consists of R-modules V_σ for every simplex $\sigma \subset \mathcal{T}$, along with linear restriction maps $r_\sigma^\tau : V_\tau \to V_\sigma$ for every inclusion $\sigma \subset \tau$, such that

- $r_\sigma^\sigma = id_{V_\sigma}$ for every σ
- For any $\sigma \subset \tau \subset \rho$, one has $r_\sigma^\tau \circ r_\tau^\rho = r_\sigma^\rho$.

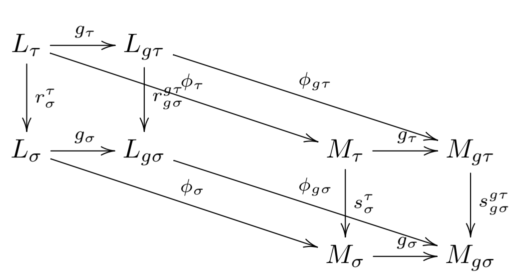
Equivalently, it is a functor from the category of simplices in $\mathcal{T}$ (with inclusions as morphisms) to the category of R-modules.

Definition 5.2. Let $\mathcal{V} = (\{V_\sigma\}_{\sigma \subset \mathcal{T}}, \{r_\sigma^\tau\}_{\sigma \subset \tau})$ be an R-coefficient system on $\mathcal{T}$. We say that $\mathcal{V}$ is *G-equivariant* if for every $g \in G$ and every simplex $\sigma \subset \mathcal{T}$, we have linear maps $g_\sigma : V_\sigma \to V_{g\sigma}$ satisfying the following properties:

- For every $g, h \in G$ and every simplex $\sigma \subset \mathcal{T}$, we have $(gh)_\sigma = g_{h\sigma} \cdot h_\sigma$
- For every simplex $\sigma \subset \mathcal{T}$, we have $1_\sigma = id_{V_\sigma}$.
- For every $g \in G$ and every inclusion $\sigma \subset \tau$, the following diagram commutes:

$$\begin{array}{ccc} V_\tau & \xrightarrow{g_\tau} & V_{g\tau} \\ \downarrow{\scriptstyle r_\sigma^\tau} & & \downarrow{\scriptstyle r_{g\sigma}^{g\tau}} \\ V_\sigma & \xrightarrow{g_\sigma} & V_{g\sigma} \end{array}$$

Definition 5.3. Let $\mathcal{L} = (\{L_\sigma\}_{\sigma \subset \mathcal{T}}, \{r_\sigma^\tau\}_{\sigma \subset \tau})$, $\mathcal{M} = (\{M_\sigma\}_{\sigma \subset \mathcal{T}}, \{s_\sigma^\tau\}_{\sigma \subset \tau})$ be G-equivariant R-coefficient systems on $\mathcal{T}$. A *morphism of G-equivariant R-coefficient systems* on $\mathcal{T}$, $\phi : \mathcal{L} \to \mathcal{M}$ consists for any simplex $\sigma \subset \mathcal{T}$ of an R-morphism $\phi_\sigma : L_\sigma \to M_\sigma$, such that for any $\sigma \subset \tau$ and any $g \in G$ the following diagram commutes



That is, for any $g \in G$ and any $\sigma \subset \mathcal{T}$ we have $\phi_{g\sigma} \circ g_\sigma = g_\sigma \circ \phi_\sigma$, and for any $\sigma \subset \tau$ we have $s_\sigma^\tau \circ \phi_\tau = \phi_\sigma \circ r_\sigma^\tau$.

The G-equivariant R-coefficient systems on $\mathcal{T}$, with the above morphisms form a category, which we denote by $\mathfrak{Coeff}_{RG}$.

Definition 5.4. Let $\mathcal{V}$ be a G-equivariant R-coefficient system on $\mathcal{T}$. Let $\widehat{\mathcal{T}_1}$ be the set of oriented edges of $\mathcal{T}$.

- The R-module $C_0(\mathcal{V})$ of *0-chains* is the set of functions $\phi : \mathcal{T}_0 \to \prod_{\sigma \in \mathcal{T}_0} V_\sigma$ with finite support such that $\phi(\sigma) \in V_\sigma$ for any vertex σ.
- The R-module $C_1(\mathcal{V})$ of *oriented 1-chains* is the set of functions $\omega : \widehat{\mathcal{T}_1} \to \prod_{\tau \in \mathcal{T}_1} V_\tau$ with finite support such that $\omega(\tau) \in V_\tau$ for any edge τ, and $\omega((\sigma, \sigma')) = -\omega((\sigma', \sigma))$. Further, denote $(\sigma', \sigma) = \overline{(\sigma, \sigma')}$.
- The *boundary map* $\partial : C_1(\mathcal{V}) \to C_0(\mathcal{V})$ is the R-linear map sending an oriented 1-chain ω supported on one edge $\tau = (\sigma, \sigma')$ to the 0-chain supported on the vertices σ, σ', with

$$\partial\omega(\sigma) = r_\sigma^\tau \omega(\sigma, \sigma'), \quad \partial\omega(\sigma') = r_{\sigma'}^\tau \omega(\sigma', \sigma)$$

Remark 5.5. The group G acts on the R-module of oriented i-chains for $i = 0, 1$, by

$$(g\omega)(g\sigma) = g(\omega(\sigma))$$

for any $g \in G$ and any oriented i-chain ω. The boundary ∂ is G-equivariant.

Definition 5.6. We define the *0-homology*

$$H_0(\mathcal{V}) = \frac{C_0(\mathcal{V})}{\partial C_1(\mathcal{V})}$$

and the *1-homology* $H_1(\mathcal{V}) = \ker \partial$. These are R-representations of G.

5.2. *Coefficient systems and stabilizers*

In what follows, let $\mathcal{V} = (\{V_\sigma\}_{\sigma \subset \mathcal{T}}, \{r_\sigma^\tau\}_{\sigma \subset \tau})$ be a G-equivariant R-coefficient system.

Proposition 5.7. *The stabilizer in G of a simplex σ acts on V_σ and the restrictions r_σ^τ, $r_{\sigma'}^\tau$ are equivariant by the intersection of the stabilizers of the vertices σ, σ' of τ.*

Proof. For any $g \in stab_G(\sigma)$, we have a linear map $g_\sigma : V_\sigma \to V_\sigma$, since $g\sigma = \sigma$. Moreover, for any $g, g' \in stab_G(\sigma)$ we have $(gg')_\sigma = g_{g'\sigma} g'_\sigma = g_\sigma g'_\sigma$, showing that we have a left action of $stab_G(\sigma)$ on V_σ. Next, let $g \in stab_G(\sigma) \cap stab_G(\sigma') = stab_G(\tau)$. Then for any $v \in V_\tau$, we have

$$r_\sigma^\tau(gv) = r_\sigma^\tau(g_\tau v) = (r_{g\sigma}^{g\tau} g_\tau)(v) = (g_\sigma r_\sigma^\tau)(v) = g_\sigma \cdot r_\sigma^\tau(v) = g \cdot r_\sigma^\tau(v)$$

showing equivariance of r_σ^τ, and similarly for $r_{\sigma'}^\tau$ ∎

Recall Definition 2.12 of the fundamental simplices v_0, v_1, e_{01}.

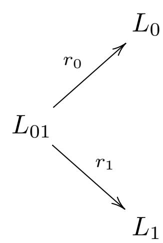
By properties of the G-action on T (see Corollary 2.13), for any oriented edge (v, v') there exists $g \in G$ such that either $g(v, v') = (v_0, v_1)$ or $g(v', v) = (v_0, v_1)$.

5.3. Diagrams

Definition 5.8. Let R be a commutative ring. An *RG-diagram* consists of the following data:

- A representation of I on an R-module L_{01}.
- A representation of K_0 on an R-module L_0.
- A representation of K_1 on an R-module L_1.
- RI-equivariant maps $r_0 : L_{01} \to L_0$, $r_1 : L_{01} \to L_1$.

We will refer to a diagram as a quintuple $(L_{01}, L_0, L_1, r_0, r_1)$, and depict such a diagram as

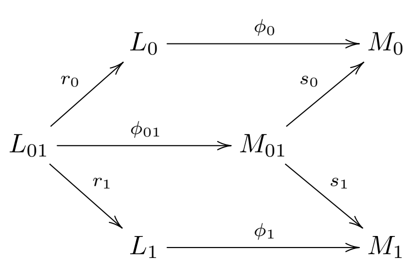


Remark 5.9. The word "diagram" was introduced by Paskunas (Paskunas [20]) in his construction of supersingular irreducible representations of $GL_2(F)$ on finite fields of characteristic p.

Definition 5.10. Let $D_1 = (L_{01}, L_0, L_1, r_0, r_1)$, $D_2 = (M_{01}, M_0, M_1, s_0, s_1)$ be R-diagrams. A *morphism of RG-diagrams* $\phi : D_1 \to D_2$ consists of the following data:

- An RI-equivariant map $\phi_{01} : L_{01} \to M_{01}$
- An RK_0-equivariant map $\phi_0 : L_0 \to M_0$
- An RK_1-equivariant map $\phi_1 : L_1 \to M_1$

such that $\phi_0 \circ r_0 = s_0 \circ \phi_{01}$ and $\phi_1 \circ r_1 = s_1 \circ \phi_{01}$, i.e. the following diagram commutes



The RG-diagrams with the above morphisms form a category, which we denote by $\mathfrak{Diag}_{RG}$.

5.4. Coefficient systems and induced representations

The coefficient systems on the tree will be intimately connected to certain representations of G. For this reason we introduce the following definition.

Proposition 5.11. *There is an equivalence of categories between RG-diagrams and G-equivariant R-coefficient systems, that is* $\mathfrak{Diag}_{RG} \simeq \mathfrak{Coeff}_{RG}$.

Proof. Consider the functor $F : \mathfrak{Coeff}_{RG} \to \mathfrak{Diag}_{RG}$ defined by

$$F(\mathcal{L}) = (L_{e_{01}}, L_{v_0}, L_{v_1}, r_{v_0}^{e_{01}}, r_{v_1}^{e_{01}}), \quad F(\phi) = (\phi_{e_{01}}, \phi_{v_0}, \phi_{v_1})$$

for any G-equivariant R-coefficient system $\mathcal{L} = (\{L_\sigma\}_{\sigma\subset\mathcal{T}}, \{r_\sigma^\tau\}_{\sigma\subset\tau})$, and any $\phi = \{\phi_\sigma\}_{\sigma\subset\tau} \in Mor_{\mathfrak{Coeff}_{RG}}(\mathcal{L}, \mathcal{M})$.

Note that indeed, as K_0, K_1, I are the stabilizers of v_0, v_1, e_{01}, respectively, since $\mathcal{L}$ is G-equivariant, we see that $L_{e_{01}}$ is an RI-module, L_{v_0} is an RK_0-module and L_{v_1} is an RK_1-module. In addition, it follows that the maps $r_{v_0}^{e_{01}}, r_{v_1}^{e_{01}}$ are RI-equivariant. It follows that $F(\mathcal{L}) \in Ob(\mathfrak{Diag}_{RG})$. Also, by definition of the morphisms we see that

$$\phi_{v_0} \circ r_{v_0}^{e_{01}} = s_{v_0}^{e_{01}} \circ \phi_{e_{01}}, \quad \phi_{v_1} \circ r_{v_1}^{e_{01}} = s_{v_1}^{e_{01}} \circ \phi_{e_{01}}$$

so that $F(\phi) \in Mor_{\mathfrak{Diag}_{RG}}(\mathcal{L}, \mathcal{M})$.

Conversely, let $D = (L_{01}, L_0, L_1, r_0, r_1)$ be an RG-diagram. Let $V_i = ind_{K_i}^G(L_i)$ and let $V_{01} = ind_I^G(L_{01})$.

Since we have embeddings $L_0 \subset V_0$, $L_1 \subset V_1$ and $L_{01} \subset V_{01}$, we think of the L_i's as embedded in the V_i's, so that we have a G-action there, and we can consider gL_i.

Recall that $\mathcal{T}_0 = \mathcal{T}_0^0 \coprod \mathcal{T}_0^1$ and for any $v \in \mathcal{T}_0^i$, there exists $g \in G$ such that $v = gv_i$. We set $L_{gv_i} = gL_i \subset V_i$. For any $e \in \mathcal{T}_1$, there exists $g \in G$ such that $e = ge_{01}$ and we set $L_{ge_{01}} = gL_{01} \subset V_1$.

Note that this is well defined, since stabilizers of the v_i act on L_i, and the stabilizer of e_{01} acts on L_{01}.

Next, we desribe the G-action. If $\sigma \subset \mathcal{T}$, either $\sigma = ge_{01}$ or $\sigma = gv_i$ for $i \in \{0,1\}$. Then either $L_{g'\sigma} = L_{g'ge_{01}} = (g'g)L_{e_{01}} = g'L_\sigma$ or $L_{g'\sigma} = L_{g'gv_i} = (g'g)L_{v_i} = g'L_\sigma$, so that the G-action is induced by its action on V_0, V_1 and V_{01}, as G preserves types.

Also, if $\sigma \subsetneq \tau$, then $\tau \in \mathcal{T}_1$, and $\sigma \in \mathcal{T}_0$. In such a case, if $\sigma \in \mathcal{T}_0^i$, there exists $g \in G$ such that $\tau = ge_{01}$ and $\sigma = gv_i$.

We set $r_\sigma^\tau = r_{gv_i}^{ge_{01}} = gr_ig^{-1}$, where the G-action is induced by its action on V_{01} and V_i. Again, this is well-defined, since r_i are RI-equivariant, and I is the stabilizer of e_{01}.

We claim that $\mathcal{L} = H(D) = (\{L_\sigma\}_{\sigma \subset \mathcal{T}}, \{r_\sigma^\tau\}_{\sigma \subset \tau})$ is a G-equivariant R-coefficient system.

Indeed, since G acts on V_0, V_1, V_{01}, we have $(gh)_\sigma = g_{h\sigma} \cdot h_\sigma$ for all $g, h \in G$ and every simplex $\sigma \subset \mathcal{T}$, and $1_\sigma = id_{L_\sigma}$.

Furthermore, by construction the r_σ^τ satisfy $r_{g\sigma}^{g\tau} \circ g_\tau = g_\sigma \circ r_\sigma^\tau$.

Also, if $\phi : D_1 \to D_2$ is a morphism of diagrams, we set $\phi_{gv_i} = g_{v_i} \phi_i \left(g^{-1}\right)_\sigma$ if $\sigma = gv_i \in \mathcal{T}_0^i$ and $\phi_\sigma = g_{e_{01}} \phi_{01} (g^{-1})_\sigma$ if $\sigma = ge_{01} \in \mathcal{T}_1$.

We then let $H(\phi) = \{\phi_\sigma\}_{\sigma \subset \mathcal{T}}$, and as $\phi_{g\sigma} \circ g_\sigma = g_\sigma \circ \phi_\sigma$, and

$$s_{gv_i}^{ge_{01}} \circ \phi_{ge_{01}} = s_{gv_i}^{ge_{01}} \circ g_{e_{01}} \circ \phi_{01} \circ (g^{-1})_{ge_{01}} = g_{v_i} \circ s_i \circ \phi_{01} \circ (g^{-1})_{ge_{01}} =$$

$$= g_{v_i} \circ \phi_i \circ r_i \circ (g^{-1})_{ge_{01}} = \phi_{gv_i} \circ g_{v_i} \circ r_i \circ (g^{-1})_{ge_{01}} = \phi_{gv_i} \circ r_{gv_i}^{ge_{01}}$$

we see that $H(\phi) \in Mor_{\mathfrak{Coeff}_{RG}}(H(D_1), H(D_2))$.

Clearly, $F \circ H = id_{\mathfrak{Diag}_{RG}}$, as F is simply a forgetful functor. Let us shows that there is a natural isomorphism $\eta : H \circ F \to id_{\mathfrak{Coeff}_{RG}}$.

To avoid confusion, we will denote by $\mathcal{L}(g)_\sigma$ the G-action on the G-equivariant R-coefficient system $\mathcal{L}$. Consider the morphism $\eta_\mathcal{L} : (H \circ F)(\mathcal{L}) \to \mathcal{L}$ defined by

$$(\eta_\mathcal{L})_{gv_i} = \mathcal{L}(g)_{v_i} \circ ((H \circ F)(\mathcal{L}))(g^{-1})_{gv_i}$$

for $i \in \{0, 1, 01\}$. This is well defined, since for $g \in K_i$, both actions $\mathcal{L}(g)$ and $(H \circ F)(\mathcal{L})(g)$ coincide with the K_i-action on L_i. (and I for L_{01}, respectively). Moreover, this is an isomorphism, as we have an inverse given by

$$((H \circ F)(\mathcal{L}))(g)_{v_i} \circ \mathcal{L}(g^{-1})_{gv_i}$$

.

Furthermore, for any $g' \in G$ we have

$$(\eta_\mathcal{L})_{g'gv_i} \circ ((H \circ F)(\mathcal{L}))(g')_{gv_i} = \mathcal{L}(g'g)_{v_i} \circ ((H \circ F)(\mathcal{L}))(g^{-1})_{gv_i} = \mathcal{L}(g')_{v_i} \circ (\eta_\mathcal{L})_{gv_i}$$

and, denoting by $(H \circ F)(r)$ the restriction maps in $(H \circ F)(\mathcal{L})$ we also have

$$r_{gv_i}^{ge_{01}} \circ (\eta_\mathcal{L})_{ge_{01}} = \mathcal{L}(g)_{v_i} \circ r_{v_i}^{e_{01}} \circ ((H \circ F)(\mathcal{L}))(g^{-1})_{ge_{01}} = (\eta_\mathcal{L})_{gv_i} \circ (H \circ F)(r)_{gv_i}^{ge_{01}}$$

showing that $\eta_\mathcal{L}$ is indeed a morphism of coefficient systems. Finally, we see that the diagram

$$\begin{array}{ccc} (H \circ F)(\mathcal{L}) & \xrightarrow{\eta_\mathcal{L}} & \mathcal{L} \\ \downarrow{\scriptstyle (H \circ F)(\phi)} & & \downarrow{\scriptstyle \phi} \\ (H \circ F)(\mathcal{M}) & \xrightarrow{\eta_\mathcal{M}} & \mathcal{M} \end{array}$$

commutes for any $\phi : \mathcal{L} \to \mathcal{M}$. Indeed, for any $\sigma = gv_i \subset \mathcal{T}$ we have

$$\phi_\sigma \circ (\eta_\mathcal{L})_\sigma = \phi_{gv_i} \circ \mathcal{L}(g)_{v_i} \circ ((H \circ F)(\mathcal{L}))(g^{-1})_{gv_i} =$$

$$= \mathcal{M}(g)_{v_i} \circ \phi_{v_i} \circ ((H \circ F)(\mathcal{L}))(g^{-1})_{gv_i} = \mathcal{M}(g)_{v_i} \circ (H \circ F)(\phi)_{v_i} \circ ((H \circ F)(\mathcal{L}))(g^{-1})_{gv_i} =$$

$$= \mathcal{M}(g)_{v_i} \circ ((H \circ F)(\mathcal{M}))(g^{-1})_{gv_i} \circ (H \circ F)(\phi)_{gv_i} = (\eta_\mathcal{M})_\sigma \circ (H \circ F)(\phi)_\sigma$$

showing that η is a natural isomorphism. $\square$

The above categorical equivalence in fact suggests the following interpretation of chains as induced representations.

Lemma 5.12. *Let $\mathcal{L}$ be a G-equivariant R-coefficient system on $\mathcal{T}$. Let $L_0 = \mathcal{L}(v_0)$, $L_1 = \mathcal{L}(v_1)$, and $L_{01} = \mathcal{L}(e_{01})$. Then*

$$C_0(\mathcal{L}) \simeq ind_{K_0}^G L_0 \oplus ind_{K_1}^G L_1, \quad C_1(\mathcal{L}) \simeq ind_I^G L_{01}$$

as RG-modules. Here $ind_H^G M$ denotes the compact induction of the $R[H]$-module M.

Proof. Let $k \in \{0,1\}$. For any pointed k-simplex $\sigma \subset \widehat{\mathcal{T}}_k$, and any k-chain $\omega \in C_k(\mathcal{L})$, we let $f_{\sigma,\omega} : G \to L_\sigma$ be the function defined by $f_{\sigma,\omega}(g) = g_{g^{-1}\sigma}\omega(g^{-1}\sigma)$. Let $K = stab_G(\sigma) = \{g \in G \mid g\sigma = \sigma\}$. Then for any $k \in K$

$$f_{\sigma,\omega}(kg) = (kg)_{g^{-1}\sigma}\omega((kg)^{-1}\sigma) = k_\sigma g_{g^{-1}\sigma}\omega(g^{-1}k^{-1}\sigma) =$$

$$= k_\sigma \cdot g_{g^{-1}\sigma}\omega(g^{-1}\sigma) = k \cdot f_{\sigma,\omega}(g)$$

Furthermore, ω has finite support, say $S = supp(\omega) \subset \mathcal{T}_k$. Let $S \cap G\sigma = \{g_\alpha \sigma \mid \alpha \in A\}$ be the elements of S in the orbit of σ under G, where $g_\alpha \in G$ are some elements. Then A is finite.
It follows that for $g \notin \bigcup_{\alpha \in A} K g_\alpha^{-1}$ one has $g^{-1} \notin \bigcup_{\alpha\in A} g_\alpha K$, hence $g^{-1}\sigma \notin S$. Indeed, if $g^{-1}\sigma \in S$, then there exists $\alpha \in A$ with $g^{-1}\sigma = g_\alpha\sigma$, hence $g_\alpha^{-1}g^{-1}\sigma = \sigma$ and $g_\alpha^{-1}g^{-1} \in K$ so that $g^{-1} \in g_\alpha K$, contradiction.
It follows that

$$f_{\sigma,\omega}(g) = g_{g^{-1}\sigma}\omega(g^{-1}\sigma) = g_{g^{-1}\sigma}(0) = 0$$

Therefore $f_{\sigma,\omega}$ is finitely supported modulo K, hence $f_{\sigma,\omega} \in ind_K^G L_\sigma$.
In addition, one has, under the G-action on $ind_K^G L_\sigma$, for any $g, h \in G$

$$(gf_{\sigma,\omega})(h) = f_{\sigma,\omega}(hg) = (hg)_{(hg)^{-1}\sigma}\omega((hg)^{-1}\sigma) =$$

$$= h_{h^{-1}\sigma}g_{g^{-1}h^{-1}\sigma}\omega(g^{-1}h^{-1}\sigma) = h_{h^{-1}\sigma}(g\omega)(h^{-1}\sigma) = f_{\sigma,g\omega}(h)$$

so that $gf_{\sigma,\omega} = f_{\sigma,g\omega}$ for any $g \in G$.
Let us consider the maps $\varphi_0 : C_0(\mathcal{L}) \to ind_{K_0}^G L_0 \oplus ind_{K_1}^G L_1$ and $\psi_0 : ind_{K_0}^G L_0 \oplus ind_{K_1}^G L_1 \to C_0(\mathcal{L})$, defined by

$$\varphi_0(\omega) = (f_{v_0,\omega}, f_{v_1,\omega}), \quad \psi_0(f_0,f_1) = \sigma \mapsto \begin{cases} g_{v_0}(f_0(g^{-1})) & \sigma = gv_0 \\ g_{v_1}(f_1(g^{-1})) & \sigma = gv_1 \end{cases}$$

For any $g \in G$ one has

$$\varphi_0(g\omega) = (f_{v_0,g\omega}, f_{v_1,g\omega}) = (gf_{v_0,\omega}, gf_{v_1,\omega}) = g(f_{v_0,\omega}, f_{v_1,\omega}) = g\varphi_0(\omega)$$

Therefore φ_0 is G-equivariant.

Conversely, if $f_0 \in ind_{K_0}^G L_0$ and $f_1 \in ind_{K_1}^G L_1$, then as they are compactly supported modulo K_0, K_1, respectively, there exist finitely many elements in G, $\{g_\alpha\}_{\alpha\in A}$, $\{g_\beta\}_{\beta\in B}$ such that

$$supp(f_0) \subset \bigcup_{\alpha\in A} K_0 g_\alpha, \quad supp(f_1) \subset \bigcup_{\beta\in B} K_1 g_\beta$$

Let $S = \{g_\alpha^{-1}\sigma_0 \mid \alpha \in A\} \cup \{g_\beta^{-1}\sigma_1 \mid \beta \in B\}$, and let $\sigma \in \mathcal{T}_0$ be such that $\sigma \notin S$. If $\sigma \in \mathcal{T}_0^0$, then there exists $g \in G$ such that $\sigma = gv_0$. As $\sigma \notin S$, we must have $gv_0 \neq g_\alpha^{-1}v_0$ for any $\alpha \in A$, so that $g^{-1}g_\alpha^{-1} \notin K_0$, hence $g^{-1} \notin K_0 g_\alpha$ for any $\alpha \in A$. It follows that $g^{-1} \notin \bigcup_{\alpha\in A} K_0 g_\alpha$, and in particular $g^{-1} \notin supp(f_0)$. Thus

$$\psi_0(f_0, f_1)(\sigma) = \psi_0(f_0, f_1)(gv_0) = g_{v_0}(f_0(g^{-1})) = g_{v_0}(0) = 0$$

Similarly, if $\sigma \in \mathcal{T}_0^1$, then $\sigma = gv_1$ for some $g \in G$ with $g^{-1} \notin \cup_{\beta\in B} K_1 g_\beta$, hence $g^{-1} \notin supp(f_1)$, so that

$$\psi_0(f_0, f_1)(\sigma) = \psi_0(f_0, f_1)(gv_1) = g_{v_1}(f_1(g^{-1})) = g_{v_1}(0) = 0$$

In any case, we see that $supp(\psi_0(f_0, f_1)) \subset S$ is finite. In addition, for any $\sigma \in \mathcal{T}_0$, if $\sigma \in \mathcal{T}_0^i$ for some $i \in \{0, 1\}$, let $g \in G$ be such that $\sigma = gv_i$, then

$$\psi_0(f_0, f_1)(\sigma) = g_{v_i}(f_i(g^{-1})) \in g_{v_i} L_i = L_{gv_i} = L_\sigma$$

Therefore, we indeed see that $\psi_0(f_0, f_1) \in C_0(\mathcal{L})$. In addition, for any $g \in G$, we see that for $\sigma = hv_i$, $i \in \{0, 1\}$ and $h \in G$ one has

$$\psi_0(g(f_0, f_1))(\sigma) = \psi_0(gf_0, gf_1)(\sigma) = h_{v_i}((gf_i)(h^{-1})) =$$

$$= h_{v_i}(f_i(h^{-1}g)) = g_{g^{-1}hv_i} \cdot (g^{-1}h)_{v_i}(f_i((g^{-1}h)^{-1})) =$$

$$= g_{g^{-1}hv_i} \cdot \psi_0(f_0, f_1)(g^{-1}hv_i) = g(\psi_0(f_0, f_1)(g^{-1}\sigma)) = (g \cdot \psi_0(f_0, f_1))(\sigma)$$

Thus, $\psi_0(g(f_0, f_1)) = g \cdot \psi_0(f_0, f_1)$, showing that ψ_0 is G-equivariant.

Furthermore, for any $\omega \in C_0(\mathcal{L})$ and any $f_0 \in ind_{K_0}^G L_0$, $f_1 \in ind_{K_1}^G L_1$ we have for any $\sigma = gv_i \in \mathcal{T}_0^i$ (here $i \in \{0, 1\}$)

$$\psi_0(\varphi_0(\omega))(\sigma) = g_{v_i}((\varphi_0(\omega))_i(g^{-1})) = g_{v_i}(f_{v_i,\omega}(g^{-1})) =$$

$$= g_{v_i}\left(\left(g^{-1}\right)_{gv_i} \cdot \omega(gv_i)\right) = \omega(gv_i) = \omega(\sigma)$$

and for any $g \in G$

$$f_{v_i,\psi_0(f_0,f_1)}(g) = g_{g^{-1}v_i}\psi_0(f_0, f_1)(g^{-1}v_i) = g_{g^{-1}v_i} \cdot \left(g^{-1}\right)_{v_i}(f_i(g)) = f_i(g)$$

Therefore $f_{v_i,\psi_0(f_0,f_1)} = f_i$ and

$$\varphi_0(\psi_0(f_0, f_1)) = (f_{v_0,\psi_0(f_0,f_1)}, f_{v_1,\psi_0(f_0,f_1)}) = (f_0, f_1)$$

This shows that φ_0, ψ_0 define isomorphisms of RG-modules

$$C_0(\mathcal{L}) \simeq ind_{K_0}^G L_0 \oplus ind_{K_1}^G L_1$$

Next, let us consider the maps $\varphi_1 : C_1(\mathcal{L}) \to ind_I^G(L_{01})$ and $\psi_1 : ind_I^G L_{01} \to C_1(\mathcal{L})$, defined by

$$\varphi_1(\omega) = f_{e_{01},\omega}, \quad \psi_1(f) = \sigma \mapsto \begin{cases} g_{e_{01}}(f(g^{-1})) & \sigma = ge_{01} \\ -g_{e_{01}}(f(g^{-1})) & \overline{\sigma} = ge_{01} \end{cases}$$

We have already shown that $f_{e_{01},\omega} \in ind_I^G(L_{01})$. Conversely, if $f \in ind_I^G L_{01}$, it is compactly supported modulo I, hence there exist finitely many elements in G, say $\{g_\alpha\}_{\alpha \in A}$ such that $supp(f) \subset \bigcup_{\alpha \in A} Ig_\alpha$.

Let $S = \{g_\alpha^{-1} e_{01}\}_{\alpha \in A}$, and let $\sigma \in \widehat{\mathcal{T}_1}$ be such that $\sigma, \overline{\sigma} \notin S$. Since G acts transitively on the set of non-oriented edges $\mathcal{T}_1$, there exists $g \in G$ such that either $\sigma = ge_{01}$ or $\overline{\sigma} = ge_{01}$.

As $\sigma, \overline{\sigma} \notin S$, we must have $ge_{01} \neq g_\alpha^{-1} e_{01}$ for any $\alpha \in A$, so that $g^{-1} g_\alpha^{-1} \notin I$, hence $g^{-1} \notin Ig_\alpha$, for any $\alpha \in A$. In particular $g^{-1} \notin supp(f)$. Therefore

$$\psi_1(f)(\sigma) = \begin{cases} \psi_1(f)(ge_{01}) & \sigma = ge_{01} \\ \psi_1(f)(\overline{ge_{01}}) & \overline{\sigma} = ge_{01} \end{cases} =$$

$$= \begin{cases} g_{e_{01}}(f(g^{-1})) & \sigma = ge_{01} \\ -g_{e_{01}}(f(g^{-1})) & \overline{\sigma} = ge_{01} \end{cases} = \begin{cases} g_{e_{01}}(0) & \sigma = g_{e_{01}} \\ -g_{e_{01}}(0) & \overline{\sigma} = g_{e_{01}} \end{cases} = 0$$

It follows that $supp(\psi_1(f)) \subset S$, hence ψ_1 is finitely supported. In addition, for any $\sigma \in \widehat{\mathcal{T}_1}$, let $g \in G$ be such that $\sigma = ge_{01}$ or $\overline{\sigma} = ge_{01}$, then

$$\psi_1(f)(\sigma) = \begin{cases} g_{e_{01}}(f(g^{-1})) & \sigma = ge_{01} \\ -g_{e_{01}}(f(g^{-1})) & \overline{\sigma} = ge_{01} \end{cases} \in g_{e_{01}} L_{01} = L_{ge_{01}} = L_\sigma$$

Also, by definition $\psi_1(f)(\overline{\sigma}) = -\psi_1(f)(\sigma)$, hence $\psi_1(f) \in C_1(\mathcal{L})$.

For any $g, h \in G$ one has

$$\varphi_1(g\omega) = f_{e_{01},g\omega} = g \cdot f_{e_{01},\omega} = g \cdot \varphi_1(\omega)$$

$$\psi_1(g \cdot f)(he_{01}) = h_{e_{01}}((gf)(h^{-1})) = h_{e_{01}}(f(h^{-1}g)) =$$

$$= g_{g^{-1}he_{01}} \cdot (g^{-1}h)_{e_{01}} \cdot f((g^{-1}h)^{-1}) = g_{g^{-1}he_{01}} \cdot \psi_1(f)(g^{-1}he_{01}) = (g \cdot \psi_1(f))(he_{01})$$

showing that both φ_1 and ψ_1 are G-equivariant.

Furthermore, for any $\omega \in C_1(\mathcal{L})$ and any $f \in ind_I^G L_{01}$, we have for any $\sigma = ge_{01} \in \widehat{\mathcal{T}_1}$ that

$$\psi_1(\varphi_1(\omega))(\sigma) = g_{e_{01}}((\varphi_1(\omega))(g^{-1})) = g_{e_{01}}(f_{e_{01},\omega}(g^{-1})) =$$

$$= g_{e_{01}}\left(\left(g^{-1}\right)_{ge_{01}} \cdot \omega(ge_{01})\right) = \omega(ge_{01}) = \omega(\sigma)$$

and for any $g \in G$

$$f_{e_{01},\psi_1(f)}(g) = g_{g^{-1}e_{01}}\psi_1(f)(g^{-1}e_{01}) = g_{g^{-1}e_{01}} \cdot \left(g^{-1}\right)_{e_{01}}(f(g)) = f(g)$$

Therefore $f_{e_{01},\psi_1(f)} = f$ and

$$\varphi_1(\psi_1(f)) = f_{e_{01},\psi_1(f)} = f$$

This shows that φ_1, ψ_1 define isomorphisms of RG-modules

$$C_1(\mathcal{L}) \simeq ind_I^G L_{01}$$

□

Lemma 5.13. *Let $\mathcal{L}$ be a G-equivariant R-coefficient system on $\mathcal{T}$. Under the isomorphisms in Lemma 5.12, the boundary map $\partial : C_1(\mathcal{L}) \to C_0(\mathcal{L})$ is described as*

$$\partial([1, l_{01}]) = ([1, r_0(l_{01})], [1, -r_1(l_{01})])$$

Here, we recall the definition of $[g, v]$, as in (3.3).

Proof. Since $ind_I^G L_{01}$ consists of functions compactly supported mod H, and G acts by right translations, we see that it is spanned, as an RG-module by elements of the form $[1, l]$. Therefore, it suffices to describe $\partial([1, l_{01}])$.

However, using the morphism ψ_1 in Lemma 5.12, we see that for any $\sigma \in \widehat{\mathcal{T}_1}$

$$\psi_1([1, l_{01}])(\sigma) = \begin{cases} g_{e_{01}}([1, l_{01}])(g^{-1}) & \sigma = ge_{01} \\ -g_{e_{01}}([1, l_{01}])(g^{-1}) & \overline{\sigma} = ge_{01} \end{cases} =$$

$$= \begin{cases} g_{e_{01}}(g^{-1} \cdot l_{01}) & \sigma = ge_{01}, g \in I \\ -g_{e_{01}}(g^{-1} \cdot l_{01}) & \overline{\sigma} = ge_{01}, g \in I \\ 0 & g \notin I \end{cases} = \begin{cases} l_{01} & \sigma = e_{01} \\ -l_{01} & \overline{\sigma} = e_{01} \\ 0 & \sigma, \overline{\sigma} \neq e_{01} \end{cases}$$

So $[1, l_{01}]$ corresponds to the 1-chain supported on the single edge e_{01}, hence $\partial(\psi_1([1, l_{01}]))$ is supported on the vertices v_0, v_1 and satisfies

$$\partial(\psi_1([1, l_{01}]))(v_0) = r_{v_0}^{e_{01}}(l_{01}) = r_0(l_{01})$$

$$\partial(\psi_1([1, l_{01}]))(v_1) = r_{v_1}^{e_{01}}(-l_{01}) = -r_1(l_{01})$$

Finally, we see that

$$\partial([1, l_{01}]) = \varphi_0(\partial(\psi_1([1, l_{01}]))) = ([1, r_0(l_{01})], [1, -r_1(l_{01})])$$

□

Corollary 5.14. *The boundary map from the oriented 1-chains to the 0-chains gives an exact sequence of RG-modules*

$$0 \to H_1(\mathcal{L}) \to ind_I^G L_{01} \to ind_{K_0}^G L_0 \oplus ind_{K_1}^G L_1 \to H_0(\mathcal{L}) \to 0$$

6. Coefficient systems and Integrality

Let F, E and G be as before. Let C be a local non archimedean field of characteristic 0, with residual field k_C of characteristic p. Let V be an irreducible locally algebraic C-representation of G.

Then by (Schneider et al. [23], Appendix, Thm 1), $V = V_{sm} \otimes_C V_{alg}$, where V_{sm} is a uniquely determined irreducible smooth representation and V_{alg} is a uniquely determined algebraic one.

When F is not contained in C, in particular when the characteristic of F is p, we make the assumption that V_{alg} is trivial. We will present a local integrality criterion for $V_{sm} \otimes V_{alg}$, by a purely representation theoretic method, not relying on the theory of (ϕ, Γ)-modules, or on rigid analytic geometry. The idea, due to Vigneras (Vignéras [29]) is to realise $V_{sm} \otimes V_{alg}$ as the 0-homology of a G-equivariant coefficent system on the tree.

We first establish some results concerning the coefficient systems on the tree. These will be used to formulate a criterion for integrality.

6.1. Coefficient systems on the tree

Let $\mathcal{T}$ be the Bruhat-Tits tree of G, and let $\mathcal{V} = (\{V_\sigma\}_{\sigma \subset \mathcal{T}}, \{r_\sigma^\tau\}_{\sigma \subset \tau})$ be a G-equivariant R-coefficient system on $\mathcal{T}$.

Definition 6.1. The *combinatorial distance* on $\mathcal{T}$ is the number of edges between two vertices. If $v, v' \in \mathcal{T}_0$, we denote it by $d(v, v')$.

Remark 6.2. This is well defined, as $\mathcal{T}$ is a tree. In fact, this is well defined also for buildings in general, as explained in (Tits [27]). The following simple lemma also holds in the general setting, but since our case is quite trivial, we prove it here as well.

Lemma 6.3. *The action of the group G respects the distance.*

Proof. Since for any two lattices L, L', and any $g \in G$, we have $L \subseteq L'$ if and only if $gL \subseteq gL'$, we see that v_1, v_2 are adjacent if and only if gv_1, gv_2 are adjacent. By induction, since $\mathcal{T}$ is a tree, we deduce the proposition for arbitrary distances. □

Definition 6.4. For any integer $n \geq 0$, we denote by S_n the sphere of vertices at distance n from v_0 and by B_n the ball of radius n. For any chain $\omega \neq 0$, let $n(\omega)$ be the integer such that the support of ω is contained in the ball $B_{n(\omega)}$ and not in $B_{n(\omega)-1}$. That is, we define

$$B_n = \{v \in \mathcal{T}_0 \mid d(v, v_0) \leq n\}, \quad S_n = \{v \in \mathcal{T}_0 \mid d(v, v_0) = n\} = B_n \backslash B_{n-1}$$

and for $\omega \in C_i(\mathcal{V})$, we set $n(\omega) = \min\{n \in \mathbb{Z} \mid \mathrm{supp}(\omega) \subseteq B_n\}$.

Remark 6.5. When ω is a 1-chain we have $n(\omega) \geq 1$.

Lemma 6.6. *For any vertex $v \in S_n$ with $n \geq 1$, the neighbours of v belong to S_{n+1} except one neighbour which belongs to S_{n-1}.*

Proof. Let $w_1, w_2 \in S_{n-1}$ be neighbours of v. Then by definition, there exist paths with no backtracking $P_1 = (v_0, \dots, w_1)$ and $P_2 = (v_0, \dots, w_2)$ of length $n-1$.

Furthermore, they do not intersect, since if $P_1 \cap P_2 \neq \emptyset$, take $p = \min\{q \in P_1 \mid q \in P_1 \cap P_2\}$ and obtain a cycle through v_0 and p.

It follows that $P_1 v P_2^{-1} = (v_0, \dots, w_1, v, w_2, \dots, v_0)$ is a cycle. But $\mathcal{T}$ is a tree, hence it must contain backtracking, and as P_1, P_2 are non-intersecting paths, we must have $(w_1, v) = (w_2, v)$, so that $w_1 = w_2$. □

Notation 6.7. Let τ_v be the unique oriented edge starting from v and pointing toward the origin v_0.

For any oriented 1-chain ω,

$$\partial\omega(v) = r_v^{\tau_v} \omega(\tau_v) \tag{6.1}$$

for all $v \in S_{n(\omega)}$.

For $i = 0, 1$, we identify naturally $u_i \in V_i = V_{v_i}$ with a 0-chain supported on the single vertex v_i. We then consider the natural K_i-equivariant linear map

$$w_i : V_i \to H_0(\mathcal{T}, \mathcal{V})$$

and the natural I-equivariant linear maps

$$w_0 \circ r_0 : V_{01} \to V_0 \to H_0(\mathcal{T}, \mathcal{V}), \quad w_1 \circ r_1 : V_{01} \to V_1 \to H_0(\mathcal{T}, \mathcal{V})$$

Lemma 6.8. *If both r_0, r_1 are injective, then:*

1. *the maps w_0, w_1 are injective.*
2. *$w_0 \circ r_0 = w_1 \circ r_1$ is I-equivariant.*

Proof. There is no non-zero 1-chain ω with $\partial\omega$ supported on the single vertex v_0 because $n(\omega) \geq 1$ and $\partial\omega$ is not zero on $S_{n(\omega)}$ by (6.1) because r_0, r_1 are injective. It follows that there is also no non-zero 1-chain ω with $\partial\omega$ supported on the single vertex v_1. As both w_i and r_i are I-equivariant, we have trivially the I-equivariance. Explicitly, if we consider ω, the oriented 1-chain with support e_{01} such that $\omega(v_0, v_1) = u_{01}$ for some $u_{01} \in V_{01}$, then we see that $\partial\omega$ is such that $\partial\omega(v_0) = r_0(u_{01})$ and $\partial\omega(v_1) = -r_1(u_{01})$, so that $\partial\omega = \omega_0 - \omega_1$, where ω_1 is the 0-chain with support v_1 and value $r_1(u_{01})$ and ω_0 is the 0-chain with support v_0 and value $r_0(u_{01})$. This shows that $w_1 \circ r_1 = w_0 \circ r_0$, as their difference is a boundary. □

When ∂ is injective, we must have $\ker r_0 \cap \ker r_1 = 0$. Indeed, else let $u \in \ker r_0 \cap \ker r_1$, and take ω to be the 1-chain supported on e_{01} with $\omega(e_{01}) = u$.

By the formula (6.1) the converse is slightly weaker.

Lemma 6.9. *∂ is injective if both r_0, r_1 are injective.*

Proof. Let $\omega \neq 0$ be any oriented 1-chain and let $v \in S_{n(w)}$; the edge τ_v belongs to the support of ω. By the formula (6.1), $\partial\omega(\sigma) = r_v^{\tau_v}\omega(\tau_v)$ does not vanish. Indeed, as the G-action is transitive on each type, there exists $g \in G$ such that $gv_0 = v$ or $gv_1 = v$ and $g(v_0, v_1) = \tau_v$ (or $-\tau_v$), whence either

$$0 = r_v^{\tau_v}\omega(\tau_v) = r_{gv_0}^{ge_{01}} \circ g_{e_{01}}(g_{e_{01}}^{-1}\omega(\tau_v)) = g_{v_0} \cdot r_{v_0}^{e_{01}}(g_{e_{01}}^{-1}\omega(\tau_v)) = g \cdot r_0(g^{-1}\omega(\tau_v))$$

or

$$0 = r_v^{\tau_v}\omega(\tau_v) = r_{gv_1}^{ge_{01}} \circ g_{e_{01}}(g_{e_{01}}^{-1}\omega(\tau_v)) = g_{v_1} \cdot r_{v_1}^{e_{01}}(g_{e_{01}}^{-1}\omega(\tau_v)) = g \cdot r_1(g^{-1}\omega(\tau_v))$$

But this implies (as the G-action is linear) that either $r_0(g^{-1}\omega(\tau_v)) = 0$ or $r_1(g^{-1}\omega(\tau_v)) = 0$, whence by injectivity of r_0 and r_1, $g^{-1}\omega(\tau_v) = 0$, hence $\omega(\tau_v) = 0$, contradiction. $\square$

We suppose from now on that the maps $r_i : V_{01} \to V_i$ are injective.

Proposition 6.10. *(Descent) Let $\phi \neq 0$ be a 0-chain not supported only at the origin. There exists an oriented 1-chain ω such that $n(\phi - \partial\omega) < n(\phi)$ if and only if $\phi(v) \in r_v^{\tau_v} V_{\tau_v}$ for all $v \in S_{n(\phi)}$.*

Proof. Let ω be an oriented 1-chain. By the formula (6.1), $n(\phi - \partial\omega) < n(\phi)$ is equivalent to $n(\omega) = n(\phi)$ and

$$\phi(v) = r_v^{\tau_v}\omega(\tau_v)$$

for all $v \in S_{n(\omega)}$. When the necessary condition $\phi(v) \in r_v^{\tau_v} V_{\tau_v}$ is satisfied, say $\phi(v) = r_v^{\tau_v}(v_{\tau_v})$ for some $v_{\tau_v} \in V_{\tau_v}$ for all $v \in S_{n(\phi)}$, the oriented 1-chain ω_ϕ supported on $\bigcup_{v \in S_{n(\phi)}} \tau_v$ with value v_{τ_v} on τ_v, satisfies $n(\phi - \partial\omega) < n(\phi)$. The oriented 1-chains satisfying $n(\phi - \partial\omega) < n(\phi)$ are $\omega_\phi + \omega'$ where $n(\omega') \leq n(\phi) - 1$. $\square$

When the R-module $r_0(V_{01})$ has a complement in V_0, say $V_0 = W_0 \oplus r_0(V_{01})$, and v is of type 0, then the R-module $r_v^{\tau_v}(V_{\tau_v})$ has a (non canonical) complement in V_v, say $V_v = W_v \oplus r_v^{\tau}(V_\tau)$; Similarly, for v of type 1, when the R-module $r_1(V_{01})$ has a complement in V_1, say $V_1 = W_1 \oplus r_1(V_{01})$, and v is of type 1, then the R-module $r_v^{\tau_v}(V_{\tau_v})$ has a (non canonical) complement in V_v.

We can find an oriented 1-chain ω supported on τ_v such that $(\phi - \partial\omega)(v) \in W_v$ for any $v \in S_{n(\phi)}$. By induction on $n(\phi)$, any nonzero element of $H_0(\mathcal{V})$ has a representative ϕ either supported at the origin, or such that $\phi(v) \in W_v$ for any $v \in S_{n(\phi)}$. (In fact, for all v).

6.2. Integrality local criterion

Let us first define what does it mean for a representation of G to be integral.

Proposition 6.11. *Let R be a complete discrete valuation ring of fraction field S. An S-representation V of G of countable dimension is integral if it admits a basis generating, over R, a G-stable R-submodule, L of V. L is called an R-integral structure.*

We will now use the machinery of coefficient systems on the tree, as described so far, to obtain a necessary and sufficient criterion for the integrality of the representation $H_0(\mathcal{V})$, for some coefficient systems $\mathcal{V}$.

Definition 6.12. Let R be a complete DVR, and let S be its fraction field. Let $\mathcal{L} = (\{L_\sigma\}_{\sigma \subset \mathcal{T}}, \{r_\sigma^\tau\}_{\sigma \subset \tau})$ be a G-equivariant R-coefficient system, and let$\mathcal{V} := \mathcal{L} \otimes_R S$, $r_{S,i} := r_i \otimes_R \mathrm{id}_S : V_{01} \to V_i$ for $i \in \{0,1\}$ be the corresponding G-equivariant S-coefficient system. Let $i \in \{0,1\}$. We say that a chain $\omega \in C_i(\mathcal{V})$ is *integral if* $\omega \in C_i(\mathcal{L}) \subset C_i(\mathcal{V})$.

By Lemma 6.8, the natural RK_0-equivariant map $w_0 : L_0 \to H_0(\mathcal{L})$ and the natural RK_1-equivariant map $w_1 : L_1 \to H_0(\mathcal{L})$ are both injective, and the natural map

$$w_0 \circ r_0 = w_1 \circ r_1 : L_{01} \to H_0(\mathcal{L})$$

is I-equivariant.

We use this result to formulate and prove the following criterion, which is just a slight variation of the criterion due to Vigneras (Vignéras [29]), and the proof is essentially the same.

Proposition 6.13. *1) $H_1(\mathcal{L}) = 0$ if r_0 and r_1 are both injective. Conversely, if $H_1(\mathcal{L}) = 0$, then* $\ker r_0 \cap \ker r_1 = 0$.

2) Integrality Local Criterion:

Suppose that

- R is a complete discrete valuation ring of fraction field S,

- L_0, L_1 are free R-modules of finite rank,

- r_0, r_1 are both injective,

and let $\mathcal{V} := \mathcal{L} \otimes_R S$, $r_{S,i} := r_i \otimes_R id_S : V_{01} \to V_i$ for $i \in \{0,1\}$.

Then, the map $H_0(\mathcal{L}) \to H_0(\mathcal{V})$ is injective, hence the R-module $H_0(\mathcal{L})$ is torsion-free and contains no line $S \cdot h$ for $h \in H_0(\mathcal{V})$, when the equivalent conditions are satisfied:

a) $r_{S,0}(V_{01}) \cap L_0 = r_0(L_{01})$ and $r_{S,1}(V_{01}) \cap L_1 = r_1(L_{01})$,

b) the maps $V_{01}/L_{01} \to V_0/L_0$, $V_{01}/L_{01} \to V_1/L_1$ are both injective.

c) $r_0(L_{01})$ is a direct factor of L_0 and $r_1(L_{01})$ is a direct factor of L_1, as R-modules.

Proof. of Proposition 6.13

1) By Lemma 6.9, $\partial : C_1(\mathcal{L}) \to C_0(\mathcal{L})$ is injective if $r_i : L_{01} \to L_i$ are both injective. However, $H_1(\mathcal{L}) = \ker \partial$, hence we are done.

2) As r_0, r_1 are injective, we can reduce them to inclusions $L_{01} \to L_0, L_{01} \to L_1$, and $r_{S,0}, r_{S,1}$ are inclusions $V_{01} \to V_0, V_{01} \to V_1$.

Equivalence of the properties $a), b), c)$:

Note that $V_{01} \cap L_i = L_{01} \iff \ker(V_{01} \to {}^{V_i}/_{L_i}) = L_{01} \iff {}^{L_i}/_{L_{01}}$ is torsion-free $\iff L_{01}$ is a direct factor of L_i (since L_i is a free module of finite rank over the PID R).

We now turn to prove the conclusion:

The R-module $H_0(\mathcal{L})$ embeds in the S-vector space $H_0(\mathcal{V})$ because the maps ${}^{V_{01}}/_{L_{01}} \to {}^{V_i}/_{L_i}$ are injective by b) hence $H_1\left({}^{\mathcal{V}}/_{\mathcal{L}}\right) = 0$ by 1) and the sequence $H_1\left({}^{\mathcal{V}}/_{\mathcal{L}}\right) \to H_0(\mathcal{L}) \to H_0(\mathcal{V})$ is exact.

Let h be a nonzero element of $H_0(\mathcal{L})$. Suppose that the line Sh is contained in $H_0(\mathcal{L})$. We choose a generator, x, for the unique maximal ideal in R, and choose

- a representative $\phi \in C_0(\mathcal{L})$ of h such that ϕ is supported on v_0 or such that $\phi(v) \in W_v$ for any $v \in S_{n(\phi)}$
- a vertex $v' \in S_{n(\phi)}$ such that $\phi(v') \neq 0$.
- an integer $n \geq 1$ such that $\phi(v') \notin x^n L_{v'}$.

Since $S \cdot h \subset H_0(\mathcal{L})$, there exists an *integral* oriented 1-cocycle $\omega \in C_1(\mathcal{L})$ such that $(\phi + \partial\omega)(v) \in x^n L_v$ for any vertex $v \in \mathcal{T}_0$.

We may suppose $n(\omega) \leq n(\phi)$ by the following argument.

If $n(\omega) > n(\phi)$, the formula (6.1) implies that $\omega(\tau_v) \in x^n L(\tau_v)$ for any vertex $v \in S_{n(\omega)}$ because $r_v^{\tau_v} L_{\tau_v} \cap x^n L_v = r_v^{\tau_v}(x^n L_{\tau_v})$ by a) and the injectivity of $r_v^{\tau_v}$. Let ω_{ext} be the integral oriented 1-cocycle supported on $\bigcup_{v \in S_n(\omega)} \tau_v$ and equal to ω on this set. We may replace ω by $\omega - \omega_{ext}$; as $n(\omega - \omega_{ext}) < n(\omega)$ we reduce to $n(\omega) \leq n(\phi)$ by decreasing induction.

If ϕ is supported on v_0, then $\omega = 0$ and $\phi(v_0) \in x^n L_0$ which is false.

If $n_\phi \geq 1$, we have $\phi(v) + \omega(\tau_v) \in x^n L_v$ for any $v \in S_{n(\phi)}$ by (6.1). As $\phi(v) \in W_v$ and $\omega(\tau_v) \in r_v^{\tau_v}(V_{\tau_v})$, this is impossible.

As R is a local complete PID, $H_0(\mathcal{L})$ is R-free. □

Lemma 6.14. *Let ϕ be a 0-chain supported at the single vertex v_0 and let ω be an oriented 1-chain such that $\phi + \partial\omega$ is integral. Then ϕ is integral.*

Proof. As $n(\omega) \geq 1$, the restriction of ω on $S_{n(\omega)}$ is integral by (6.1). By a decreasing induction on $n(\omega)$, ϕ is integral. □

Corollary 6.15. *When the properties of 2) (in Proposition 6.13) are true, $H_0(\mathcal{L})$ is an R-integral structure of $H_0(\mathcal{V})$ such that*

$$H_0(\mathcal{L}) \cap V_0 = L_0, \quad H_0(\mathcal{L}) \cap V_1 = L_1$$

The corollary in fact shows that the above criterion is sufficient for $H_0(\mathcal{V})$ to be integral, where $\mathcal{V} = \mathcal{L} \otimes_R S$. Next, we establish a necessary and sufficient criterion.

Corollary 6.16. *Let R be a complete discrete valuation ring of fraction field S, and let $r_0 : V_{01} \to V_0$, $r_1 : V_{01} \to V_1$ be the maps in the RG-diagram corresponding to a G-equivariant R-coefficient system $\mathcal{V}$. The S-representation $H_0(\mathcal{V})$ of G is R-integral if and only if there exist R-integral structures L_0, L_1 of the representations V_0 of K_0, V_1 of K_1, such that $L_{01} = r_0^{-1}(L_0) = r_1^{-1}(L_1)$. When this is true, the diagram*

$$\begin{array}{ccc} & & L_0 \\ & \overset{r_0}{\nearrow} & \\ L_{01} & & \\ & \underset{r_1}{\searrow} & \\ & & L_1 \end{array}$$

defines a G-equivariant coefficient system $\mathcal{L}$ of R-modules on $\mathcal{X}$, and $H_0(\mathcal{L})$ is an R-integral structure of $H_0(\mathcal{V})$.

Proof. of Corollary 6.16

Sufficient. When L_0, L_1 are R-integral structures of V_0, V_1 such that $L_{01} = r_0^{-1}(L_0) = r_1^{-1}(L_1)$, then L_{01} is an R-integral structure of V_{01}; the maps r_0, r_1 induce an injective diagram

$$\begin{array}{ccc} & & L_0 \\ & \nearrow & \\ L_{01} & & \\ & \searrow & \\ & & L_1 \end{array}$$

By the integrality criterion (Proposition 6.13), $H_0(\mathcal{V})$ is R-integral.

Necessary. Suppose that L is an R-integral structure of $H_0(\mathcal{V})$. We apply Lemma 6.8. The inverse image L_0 of $w_0(V_0) \cap L$ in V_0 by w_0 is an R-integral structure of the representation of K_0 on V_0, the inverse image L_1 of $w_1(V_1) \cap L$ in V_1 by w_1 is an R-integral structure of the representation of K_1 on V_1, and the inverse image L_{01} of $(w_0 \circ r_0)(V_{01}) \cap L = (w_1 \circ r_1)(V_{01}) \cap L$ is an R-integral structure of V_{01}, such that $L_{01} = r_0^{-1}(L_0) = r_1^{-1}(L_1)$. $\square$

From now on, r_0, r_1 are injective and $V_0 = K_0 \cdot r_0(V_{01}), \quad V_1 = K_1 \cdot r_1(V_{01})$.

Definition 6.17. When V_i, for $i = 0, 1$ identified with an element of $\mathbb{Z}/2\mathbb{Z}$, contains an R-integral structure M_i which is a finitely generated R-submodule, one constructs inductively an increasing sequence of finitely generated R-integral structures $(z^n(M_i))_{n\geq 1}$ of V_i, called the *zigzags* of M_i, as follows:

The RK_{i+1}-module M_{i+1} defined by $M_{i+1} = K_{i+1} \cdot r_{i+1}(r_i^{-1}(M_i))$ is an R-integral structure of the SK_{i+1}-module V_{i+1} (a finitely generated R-module is free if and only if it is torsion free and does not contain an S-line). We repeat this construction to get the first zigzag $z(M_i)$:

$$z(M_i) = K_i \cdot r_i \left(r_{i+1}^{-1} \left(K_{i+1} \cdot r_{i+1} \left(r_i^{-1}(M_i)\right)\right)\right) \supseteq M_i$$

Corollary 6.18. *Let $i \in \mathbb{Z}/2\mathbb{Z}$ and let M_i be an R-integral structure of the SK_i-module V_i. The representation of G on $H_0(\mathcal{V})$ is R-integral if and only if the sequence of zigzags $(z^n(M_i))_{n\geq 0}$ is finite.*

Proof. of Corollary 6.18

When the sequence of zigzags is finite, there exists a finitely generated R-integral structure M_i of V_i equal to its first zigzag $z(M_i) = M_i$, for $i = 0$ or $i = 1$. Set $M_{i+1} = r_{i+1}(r_i^{-1}(M_i))$. By definition of $z(M_i)$, we see that

$$M_i = z(M_i) = K_i \cdot r_i \left(r_{i+1}^{-1} \left(K_{i+1} \cdot M_{i+1}\right)\right) \supset r_i \left(r_{i+1}^{-1} \left(K_{i+1} \cdot M_{i+1}\right)\right)$$

hence

$$M_{i+1} = r_{i+1} \left(r_i^{-1}(M_i)\right) \supset K_{i+1} \cdot M_{i+1}$$

showing that $M_{i+1} \subset V_{i+1}$ is K_{i+1}-stable, hence a finitely generated R-integral structure of V_{i+1}, such that $r_{i+1}^{-1}(M_{i+1}) = r_i^{-1}(M_i)$.

Conversely, let M_i be an R-integral structure of V_i. Replacing L by a multiple, we suppose $M_i \subset L_i$. Then $r_i^{-1}(M_i) \subset r_i^{-1}(L_i) = r_{i+1}^{-1}(L_{i+1})$, hence $K_{i+1} \cdot r_{i+1} \left(r_i^{-1}(M_i)\right) \subset L_{i+1}$ and $z(M_i) \subset L_i$. The sequence of zigzags of M_i is contained in L_i and increasing, hence finite because L_i is a finitely generated R-module and R is noetherian. □

6.3. Integrality criterion for locally algebraic representations

The main idea allowing us to make use of the above criterion for arbitrary irreducible locally algebraic representations, is the fact that any such representation can be obtained as the 0-homology of some coefficient system on the tree. This was shown for smooth representations over $\mathbb{C}$ by Schneider and Stuhler in Schneider and Stuhler [21], and we will extend the result further here. The proof is the same as in Vignéras [29] for the case $G = GL_2(F)$.

In order to formulate the results, we will use the filtrations previously described of the stabilizers (see subsection 2.3).

Lemma 6.19. *We have group homomorphisms:*

When E/F is unramified

$$K_0/K_0(1) \simeq \mathbf{U_3}(k_F), \quad K_1/K_1(1) \simeq \mathbf{H}(k_F), \quad I/I(1) \simeq \mathbf{M}(k_F)$$

while for E/F ramified we have

$$^{K_0}/_{K_0(1)} \simeq \mathbf{O_3}(k_F), \quad ^{K_1}/_{K_1(1)} \simeq \mathbf{H}'(k_F), \quad ^{I}/_{I(1)} \simeq \mathbf{M}'(k_F)$$

and further bijections, when E/F is unramified

$$B \cap K_0 \backslash K_0/K_0(1) \simeq \mathbf{B}(k_F)\backslash \mathbf{G}(k_F)$$

$$B \cap K_1 \backslash K_1/K_1(1) \simeq \mathbf{M}(k_F)\mathbf{Z}(k_F)\backslash \mathbf{H}(k_F)$$

and when E/F is ramified

$$B \cap K_0 \backslash K_0/K_0(1) \simeq \mathbf{B}'(k_F)\backslash \mathbf{O_3}(k_F)$$

$$B \cap K_1 \backslash K_1/K_1(1) \simeq \mathbf{M}'(k_F)\mathbf{Z}'(k_F)\backslash \mathbf{H}'(k_F)$$

Here $\mathbf{G}(k_F) = \mathbf{U}_3(k_F) = \{g \in GL_3(k_E) \mid^t \overline{g}\theta g = \theta\}$ is the unitary group in three variables, $\mathbf{O}_3(k_F) = \{g \in GL_3(k_F) \mid^t g\theta g = \theta\}$ is the corresponding orthogonal group in three variables, $\mathbf{B}(k_F), \mathbf{B}'(k_F)$ are the Borel subgroups of upper triangular matrices in $\mathbf{U_3}(k_F), \mathbf{O}_3(k_F)$, respectively, $\mathbf{M}(k_F), \mathbf{M}'(k_F)$ the corresponding Levi qoutients, and

$$\mathbf{H}(k_F) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{pmatrix} \mid c \in k_E^1, \quad a\overline{d}, b\overline{e} \in k_E^-, \quad \overline{a}e + \overline{d}b = 1 \right\} \leq \mathbf{G}(k_F)$$

$$\mathbf{H}'(k_F) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & \pm 1 & 0 \\ c & 0 & d \end{pmatrix} \mid ad - bc = 1 \right\} \leq \mathbf{O_3}(k_F)$$

$$\mathbf{Z}(k_F) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid z \in k_E, \quad z + \overline{z} = 0 \right\}$$

$$\mathbf{Z}'(k_F) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid z \in k_F \right\}$$

Proof. We have natural maps $K_0 \to \mathbf{U_3}(k_F)$, $I \to \mathbf{M}(k_F)$ in the unramified case, and $K_0 \to \mathbf{O_3}(k_F)$, $I \to \mathbf{M}'(k_F)$ in the ramified case, by reducing each entry mod $\pi\mathcal{O}_E$, and zeroes above the main diagonal in the latter map. Further we have a map $\rho : K_1 \to \mathbf{H}(k_F)$ in the unramified case, or $\rho : K_1 \to \mathbf{H}'(k_F)$ in the ramified case defined by

$$\rho \begin{pmatrix} a & b & \pi^{-1}c \\ \pi d & e & f \\ \pi g & \pi h & i \end{pmatrix} = \begin{pmatrix} a \mod \pi & 0 & c \mod \pi \\ 0 & e \mod \pi & 0 \\ g \mod \pi & 0 & i \mod \pi \end{pmatrix}$$

Note that $K_0(1), I(1)$ and $K_1(1)$ are precisely the kernels of these surjective maps to obtain the required result.

For the bijections, note that $B \cap K_0$ is the preimage of $\mathbf{B}(k_F)$, $\mathbf{B}'(k_F)$ under the reduction maps $K_0 \to \mathbf{G}(k_F)$, $K_0 \to \mathbf{O_3}(k_F)$, while $B \cap K_1$ is the preimage of $\mathbf{M}(k_F)\mathbf{Z}(k_F)$, $\mathbf{M}'(k_F)\mathbf{Z}'(k_F)$ under the reduction maps $K_1 \to \mathbf{H}(k_F)$, $K_1 \to \mathbf{H}'(k_F)$. □

Proposition 6.20. *Let V_{alg} be an irreducible algebraic C-representation of G (hence $E \subset C$ if V_{alg} is not trivial), let V_{sm} be a finite length smooth C-representation of G and let e be an integer ≥ 1 such that V_{sm} is generated by its $K_0(e)$-invariants.*

1) The locally algebraic C-representation $V := V_{sm} \otimes_C V_{alg}$ of G is isomorphic to the 0*-th homology $H_0(\mathcal{V})$ of the coefficient system $\mathcal{V}$ associated with the inclusions*

$$\begin{array}{ccc} & & V_{sm}^{K_0(e)} \otimes_C V_{alg} \\ & \nearrow & \\ V_{sm}^{I(e)} \otimes_C V_{alg} & & \\ & \searrow & \\ & & V_{sm}^{K_1(e)} \otimes_C V_{alg} \end{array}$$

2) The representation of G on V is $\mathcal{O}_C$-integral if and only if there exist $\mathcal{O}_C$-integral structures L_0, L_1 of the representations of K_0, K_1 on $V_{sm}^{K_0(e)} \otimes_C V_{alg}$, $V_{sm}^{K_1(e)} \otimes_C V_{alg}$ respectively such that $L_{01} := L_0 \cap (V_{sm}^{I(e)} \otimes_C V_{alg}) = L_1 \cap (V_{sm}^{I(e)} \otimes_C V_{alg})$. Then the 0-th homology L of the G-equivariant coefficient system on $\mathcal{X}$ defined by the diagram

$$\begin{array}{ccc} & & L_0 \\ & \nearrow & \\ L_{01} & & \\ & \searrow & \\ & & L_1 \end{array}$$

is an $\mathcal{O}_C$-structure of V.

Proof. of Proposition 6.20

1) The exactness of the sequence

$$0 \to ind_I^G \left(V_{sm}^{I(e)} \otimes_C V_{alg}\right) \to ind_{K_0}^G \left(V_{sm}^{K_0(e)} \otimes_C V_{alg}\right) \oplus ind_{K_1}^G \left(V_{sm}^{K_1(e)} \otimes_C V_{alg}\right) \to$$
$$\to V_{sm} \otimes_C V_{alg} \to 0$$

follows from the following facts.

The assertion is true when V_{alg} is trivial if C is replaced by the field $\mathbb{C}$ of complex numbers by (Schneider and Stuhler [21] II.3.1); This is also true for C because the scalar extension $\otimes_C \mathbb{C}$ commutes with the invariants by an open compact subgroup and with the compact induction from an open subgroup. The tensor product by $\otimes_C V_{alg}$ of an exact sequence of CG-representations remains exact and commutes with the compact induction from an open subgroup.

2) The finite length representation V_{sm} is admissible; this is known for complex representations and remains true for C-representations because $V_{sm} \otimes_C \mathbb{C}$ has finite length (Vignéras [28] $II.43.c$), and $\cdot \otimes_C \mathbb{C}$ commutes with the $K_i(e)$-invariant functor. The C-vector spaces $V_{sm}^{K_i(e)} \otimes_C V_{alg}$ are finite dimensional. Apply Corollary 6.16. $\square$

We have $L_0 = L \cap (V_{sm}^{K_0(e)} \otimes V_{alg})$ and $L_1 = L \cap \left(V_{sm}^{K_1(e)} \otimes V_{alg}\right)$ in 2) by Lemma 6.14; when $(V_{sm}^{K_0(e)} \otimes_C V_{alg}) = K_0 \cdot (V_{sm}^{I(e)} \otimes_C V_{alg})$ and $(V_{sm}^{K_1(e)} \otimes_C V_{alg}) = K_1 \cdot (V_{sm}^{I(e)} \otimes_C V_{alg})$, one can suppose $L_0 = K_0 L_{01}$ and $L_1 = K_1 L_{01}$ in 2) by Corollary 6.18.

We define the contragredient $\tilde{V} = \tilde{V}_{sm} \otimes_C V_{alg}^{'}$ of $V = V_{sm} \otimes_C V_{alg}$ by tensoring the smooth contragredient $\tilde{V}_{sm}$ of V_{sm} and the linear contragredient $V_{alg}^{'}$ of V_{alg}.

Corollary 6.21. *A finite length locally algebraic C-representation of G is $\mathcal{O}_C$-integral if and only if its contragredient is $\mathcal{O}_C$-integral.*

Proof. of Corollary 6.21

Let V_{sm} be a nonzero smooth C-representation of G of finite length; there exists an integer $e \geq 1$ such that each nonzero irreducible subquotient of V_{sm} contains a nonzero $K_i(e)$-invariant vector, by smoothness. The C-vector space $\left(\tilde{V}_{sm}\right)^{K_i(e)}$ is isomorphic to the dual $\left(V_{sm}^{K_i(e)}\right)^{'}$; the irreducible subquotients of the contragredient $\tilde{V}_{sm}$ are the contragredients of the irreducible subquotients of V_{sm}. Hence V_{sm} and $\tilde{V}_{sm}$ are generated by their $K_0(e)$-invariants.

Suppose that $V = V_{sm} \otimes_C V_{alg}$ is $\mathcal{O}_C$-integral. We choose $\mathcal{O}_C$-integral structures L_0 of the representation of K_0 on $V_{sm}^{K_0(e)} \otimes_C V_{alg}$ and L_1 of the representation of K_1 on $V_{sm}^{K_1(e)} \otimes_C V_{alg}$ such that $L_{01} := L_1 \cap \left(V_{sm}^{I(e)} \otimes_C V_{alg}\right) = L_0 \cap \left(V_{sm}^{I(e)} \otimes_C V_{alg}\right)$ (Proposition 6.20), and we take the linear duals $L_0^{'} = Hom_{\mathcal{O}_C}(L_0, \mathcal{O}_C)$ of L_0 and $L_1^{'} = Hom_{\mathcal{O}_C}(L_1, \mathcal{O}_C)$ of L_1. It is clear that $L_i^{'}$ is an $\mathcal{O}_C$-integral structure of the representation of K_i on

$$\left(V_{sm}^{K_i(e)} \otimes_C V_{alg}\right)^{'} \simeq \left(V_{sm}^{K_i(e)}\right)^{'} \otimes_C (V_{alg})^{'} \simeq \left(\tilde{V}_{sm}\right)^{K_i(e)} \otimes_C V_{alg}^{'}$$

We take the intersection $L_i^{'} \cap \left(\left(\tilde{V}_{sm}\right)^{I(e)} \otimes_C V_{alg}^{'}\right) = L_i^{'} \cap \left(V_{sm}^{I(e)} \otimes_C V_{alg}\right)^{'}$.

The $\mathcal{O}_C$-module L_{01} is a direct factor of L_0 and L_1, hence its linear dual $L_{01}^{'}$ is equal to this intersection. By Proposition 6.20, $\tilde{V}$ is $\mathcal{O}_C$-integral. $\square$

From now on, we assume $V_{alg} = C$, i.e. the algebraic part is trivial, and the representation is smooth.

We conclude this part by considering inflations of diagrams.

Definition 6.22. A *tamely ramified diagram* is a diagram $D = (L_{01}, L_0, L_1, r_0, r_1)$ such that

- $K_0(1)$ acts trivially on L_0.
- $K_1(1)$ acts trivially on L_1.
- $I(1)$ acts trivially on L_{01}, and it is semi-simple as an SI-module.
- r_0, r_1 are injective.

Lemma 6.23. *A tamely ramified diagram is equivalent (by "inflation") to the following data if E/F is unramified:*

- an R-representation Y_0 of $\mathbf{U_3}(k_F)$.
- an R-representation Y_1 of $\mathbf{H}(k_F)$.
- a semi-simple R-representation Y_{01} of $\mathbf{M}(k_F)$.
- $R\mathbf{M}(k_F)$-inculsions $Y_{01} \to Y_0$ and $Y_{01} \to Y_1$ with images contained in $Y_0^{\mathbf{N}(k_F)}$ and $Y_1^{\mathbf{Z}(k_F)}$ respectively.

and to to the following data if E/F is ramified:

- an R-representation Y_0 of $\mathbf{O_3}(k_F)$.
- an R-representation Y_1 of $\mathbf{H}'(k_F)$.
- a semi-simple R-representation Y_{01} of $\mathbf{M}'(k_F)$.
- $R\mathbf{M}'(k_F)$-inculsions $Y_{01} \to Y_0$ and $Y_{01} \to Y_1$ with images contained in $Y_0^{\mathbf{N}'(k_F)}$ and $Y_1^{\mathbf{Z}'(k_F)}$ respectively.

The action of K_0 on L_0 inflates the action of Y_0, the action of K_1 on L_1 inflates the action of Y_1, the action of I on L_{01} inflates the action of Y_{01}.

Here,

$$\mathbf{N}'(k_F) = \left\{ n_{b,z} = \begin{pmatrix} 1 & b & z \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} \mid 2b + z^2 = 0, \quad b, z \in k_F \right\} \leq \mathbf{O_3}(k_F)$$

is its unipotent radical. If $p \neq 2$, $z = -b^2/2$, and we denote $n_b = n_{b,z}$. If $p = 2$, then $b = 0$.

Proof. This is an immediate consequence of Lemma 6.19. □

7. Integrality of principal series representations

We will make explicit the integral structures contsructed in Proposition 6.20, when $V = V_{sm}$ is a smooth tamely ramified principal series representation.

When $\chi \otimes \chi_1$ is tamely ramified, i.e. trivial on $M \cap I(1)$, its restriction to $M \cap I$ is the inflation of a C-character $\eta \otimes \eta_1$ of $\mathbf{M}(k_F)$, and the principal series is the 0-th homology of the G-equivariant coefficient system defined by the tamely ramified diagram

$$\begin{array}{ccc} & & \left(ind_B^G(\chi \otimes \chi_1)\right)^{K_0(1)} \\ & \nearrow & \\ \left(ind_B^G(\chi \otimes \chi_1)\right)^{I(1)} & & \\ & \searrow & \\ & & \left(ind_B^G(\chi \otimes \chi_1)\right)^{K_1(1)} \end{array}$$

inflated, in the case of E/F unramified, from the inclusions (see Lemma 7.2)

$$\left(ind_{\mathbf{B}(k_F)}^{\mathbf{G}(k_F)}(\eta \otimes \eta_1)\right)^{\mathbf{N}(k_F)} \to ind_{\mathbf{B}(k_F)}^{\mathbf{G}(k_F)}(\eta \otimes \eta_1)$$

$$\left(ind_{\mathbf{M}(k_F)\mathbf{Z}(k_F)}^{\mathbf{H}(k_F)}(\eta \otimes \eta_1)\right)^{\mathbf{Z}(k_F)} \to ind_{\mathbf{M}(k_F)\mathbf{Z}(k_F)}^{\mathbf{H}(k_F)}(\eta \otimes \eta_1)$$

and in the case of E/F ramified, from the inclusions (see Lemma 7.2)

$$\left(ind_{\mathbf{B}'(k_F)}^{\mathbf{O_3}(k_F)}(\eta \otimes \eta_1)\right)^{\mathbf{N}'(k_F)} \to ind_{\mathbf{B}'(k_F)}^{\mathbf{O_3}(k_F)}(\eta \otimes \eta_1)$$

$$\left(ind_{\mathbf{M}'(k_F)\mathbf{Z}'(k_F)}^{\mathbf{H}'(k_F)}(\eta \otimes \eta_1)\right)^{\mathbf{Z}'(k_F)} \to ind_{\mathbf{M}'(k_F)\mathbf{Z}'(k_F)}^{\mathbf{H}'(k_F)}(\eta \otimes \eta_1)$$

Note that $\left(ind_{\mathbf{B}(k_F)}^{\mathbf{G}(k_F)}(\eta \otimes \eta_1)\right)^{\mathbf{N}(k_F)} = C \cdot \phi_1 \oplus C \cdot \phi_s$, where ϕ_1, ϕ_s have supports $\mathbf{B}(k_F)$, $\mathbf{B}(k_F)s\mathbf{N}(k_F)$ and value 1 at id, s, respectively. Clearly,

$$Y_{01} := \mathcal{O}_C \cdot \phi_1 \oplus \mathcal{O}_C \cdot \phi_s$$

is an $\mathcal{O}_C$-integral structure of $\left(ind_{\mathbf{B}(k_F)}^{\mathbf{G}(k_F)}(\eta \otimes \eta_1)\right)^{\mathbf{N}(k_F)}$ and

$$Y_1 := \mathbf{H}(k_F) \cdot Y_{01}, \quad Y_0 := \mathbf{G}(k_F) \cdot Y_1^{\mathbf{Z}(k_F)}$$

are $\mathcal{O}_C$-integral structures of $ind_{\mathbf{M}(k_F)\mathbf{Z}(k_F)}^{\mathbf{H}(k_F)}(\eta \otimes \eta_1)$ and $ind_{\mathbf{B}(k_F)}^{\mathbf{G}(k_F)}(\eta \otimes \eta_1)$.

Similarly, note that $\left(ind_{\mathbf{B'}(k_F)}^{\mathbf{O_3}(k_F)}(\eta\otimes\eta_1)\right)^{\mathbf{N'}(k_F)} = C\cdot\phi_1\oplus C\cdot\phi_s$, where ϕ_1,ϕ_s have supports $\mathbf{B'}(k_F)$, $\mathbf{B'}(k_F)s\mathbf{N'}(k_F)$ and value 1 at id, s, respectively. Clearly,

$$Y_{01} := \mathcal{O}_C\cdot\phi_1\oplus\mathcal{O}_C\cdot\phi_s$$

is an $\mathcal{O}_C$-integral structure of $\left(ind_{\mathbf{B'}(k_F)}^{\mathbf{O_3}(k_F)}(\eta\otimes\eta_1)\right)^{\mathbf{N'}(k_F)}$ and

$$Y_1 := \mathbf{H'}(k_F)\cdot Y_{01},\quad Y_0 := \mathbf{O_3}(k_F)\cdot Y_{01}^{\mathbf{Z'}(k_F)}$$

are $\mathcal{O}_C$-integral structures of $ind_{\mathbf{M'}(k_F)\mathbf{Z'}(k_F)}^{\mathbf{H'}(k_F)}(\eta\otimes\eta_1)$ and $ind_{\mathbf{B'}(k_F)}^{\mathbf{G'}(k_F)}(\eta\otimes\eta_1)$.
Now (Y_0, Y_1, Y_{01}) inflates to a tamely ramified diagram

$$\begin{array}{ccc} & & L_{Y_0} = K_0\cdot L_{Y_{01}} \\ & \nearrow & \\ L_{Y_{01}} & & \\ & \searrow & \\ & & L_{Y_1} = K_1\cdot L_{Y_1} \end{array}$$

defining a G-equivariant coefficient system $\mathcal{L}$ of free $\mathcal{O}_C$-modules of finite rank on $\mathcal{T}$.

7.1. Integrality criterion for smooth principal series representations

In this section we will prove the following theorem -

Theorem 7.1. *Suppose that the character $\chi\otimes\chi_1$ is tamely ramified, and that C contains a p-th root of 1. The following properties are equivalent:*
a) the principal series representation $ind_B^G(\chi\otimes\chi_1)$ is $\mathcal{O}_C$-integral.
b) $\chi^{-1}(\pi)$, $\chi(\pi)q^2$ are integral.
c) $Y_0^{\mathbf{N}(k_F)} = Y_{01} = Y_1^{\mathbf{Z}(k_F)}$, when E/F is unramified, while $Y_0^{\mathbf{N'}(k_F)} = Y_{01} = Y_1^{\mathbf{Z'}(k_F)}$ when E/F is ramified.
d) $L := H_0(\mathcal{L})$ is an $\mathcal{O}_C$-integral structure of $ind_B^G(\chi\otimes\chi_1)$.
When they are satisfied, we have $L^{K_0(1)} = L_{Y_0}$, $L^{K_1(1)} = L_{Y_1}$ and $L^{I(1)} = L_{Y_{01}}$ generates the $\mathcal{O}_C G$-module L.

Note that we may reduce to the case $\chi_1 = 1$, twisting by a central character.
Hence, it suffices to assume χ tamely ramified and $\chi_1 = 1$.
As χ is a tamely ramified character, $\eta = \chi\restriction_{\mathcal{O}_E^\times}$ is the inflation of a character of $\mathbb{F}_q^\times = k_E^\times \simeq \mathcal{O}_E^\times/(1+\pi\mathcal{O}_E)$, that we denote by the same letter, η.
We will also denote $\chi(\pi) = \lambda$, so that $\chi(x) = \lambda^{v_E(x)}\cdot\eta(x\pi^{-v_E(x)})$, where v_E is the standard valuation on E (normalized such that $v_E(\pi) = 1$).

Lemma 7.2. *Let $\chi : E^\times \to C$ be a tamely ramified character. It induces a character $\eta : \mathbb{F}_q^\times \to C$. We have, when E/F is unramified,*

$$\left(ind_B^G \chi\right)^{K_0(1)} \cong ind_{\mathbf{B}(k_F)}^{\mathbf{U}_3(k_F)} \eta, \quad \left(ind_B^G \chi\right)^{K_1(1)} \cong ind_{\mathbf{M}(k_F)\mathbf{Z}(k_F)}^{\mathbf{H}(k_F)} \eta$$

and when E/F is ramified,

$$\left(ind_B^G \chi\right)^{K_0(1)} \cong ind_{\mathbf{B}'(k_F)}^{\mathbf{O}_3(k_F)} \eta, \quad \left(ind_B^G \chi\right)^{K_1(1)} \cong ind_{\mathbf{M}'(k_F)\mathbf{Z}'(k_F)}^{\mathbf{H}'(k_F)} \eta$$

Proof. As $G = BK_0 = BK_1$, by the Iwasawa decompositon (Lemma 2.17), we see that

$$\left(ind_B^G \chi\right) \restriction_{K_0} \simeq ind_{B\cap K_0}^{K_0} \eta, \quad \left(ind_B^G \chi\right) \restriction_{K_1} \simeq ind_{B\cap K_1}^{K_1} \eta$$

and the representations of K_0, K_1 on $\left(ind_B^G \chi\right)^{K_0(1)}, \left(ind_B^G \chi\right)^{K_1(1)}$, respectively, are the inflations of the principal series representations (see Lemma 6.23)

$$ind_{\mathbf{B}(k_F)}^{\mathbf{G}(k_F)} \eta, \quad ind_{\mathbf{M}(k_F)\mathbf{Z}(k_F)}^{\mathbf{H}(k_F)} \eta$$

when E/F is unramified, and inflations of the principal series representations

$$ind_{\mathbf{B}'(k_F)}^{\mathbf{O_3}(k_F)} \eta, \quad ind_{\mathbf{M}'(k_F)\mathbf{Z}'(k_F)}^{\mathbf{H}'(k_F)} \eta$$

when E/F is ramified. □

In what follows we will introduce the set-up for our proof.

We construct explicit integral structures in $V_0 = (ind_B^G \chi)^{K_0(1)}, V_1 = \left(ind_B^G \chi\right)^{K_1(1)}$, and compute their preimages in $V_{01} = \left(ind_B^G \chi\right)^{I(1)}$.

We will begin our zig-zag by considering the natural choice of an integral structure in $V_0 = \left(ind_B^G \chi\right)^{K_0(1)}$.

7.1.1. *The integral structure L_0*

Let L_0 be the $\mathcal{O}_C$-integral structure of the C-representation of K_0 on $V_0 = \left(ind_{B\cap K_0}^{K_0} \eta\right)^{K_0(1)}$ given by the functions with values in $\mathcal{O}_C$.

We denote by $f_g \in L_0$ the function of support $(B \cap K_0)\, gK_0(1)$ and value 1 at g.

When E/F is unramified, a system of representatives for $B \cap K_0 \backslash K_0/K_0(1) \simeq \mathbf{B}(k_F)\backslash\mathbf{G}(k_F)$ (see Lemma 6.19) is

$$1, sn \text{ for } n \in \mathbf{N}(k_F), \quad s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

as follows from the Bruhat decomposition $\mathbf{G}(k_F) = \mathbf{B}(k_F) \coprod \mathbf{B}(k_F)s\mathbf{N}(k_F)$.

When E/F is ramified, a system of representatives for $B \cap K_0 \backslash K_0 / K_0(1) \simeq \mathbf{B}'(k_F) \backslash \mathbf{O_3}(k_F)$ (see Lemma 6.19) is

$$1, sn \text{ for } n \in \mathbf{N}'(k_F)$$

as follows from the Bruhat decomposition $\mathbf{O_3}(k_F) = \mathbf{B}'(k_F) \coprod \mathbf{B}'(k_F) s \mathbf{N}'(k_F)$. Therefore, an $\mathcal{O}_C$-basis of L_0 is $\left\{ f_1, (f_{sn})_{n \in \mathbf{N}(k_F)} \right\}$ when E/F is unramified, and $\left\{ f_1, (f_{sn})_{n \in \mathbf{N}'(k_F)} \right\}$ when E/F is ramified. Note that for any $n_{b,z} \in \mathbf{N}(k_F)$ we have

$$n_{b,z}^{-1} = \begin{pmatrix} 1 & b & z \\ 0 & 1 & -\overline{b} \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -b & \overline{z} \\ 0 & 1 & \overline{b} \\ 0 & 0 & 1 \end{pmatrix} = n_{-b,\overline{z}}$$

Similarly, in the ramified case, $n_b^{-1} = n_{-b}$ and when $p = 2$, $n_z^{-1} = n_z$.

In what follows, if E/F is unramified, we will denote by $\rho_0 : K_0 \to \mathbf{G}(k_F)$ and $\rho_1 : K_1 \to \mathbf{H}(k_F)$ the two natural reductions, while if E/F is ramified, we will use the same notations, only that $\rho_0 : K_0 \to \mathbf{O}_3(k_F)$ and $\rho_1 : K_1 \to \mathbf{H}'(k_F)$.

The following property of L_0 will turn useful when computing its zig-zag.

Proposition 7.3. *The $\mathcal{O}_C K_0$-module L_0 is cyclic, generated by f_1, i.e. $L_0 = \mathcal{O}_C K_0 \cdot f_1$.*

Proof. For all $n \in \mathbf{N}(k_F)$, and all $g \in K_0$, one has

$$nsf_1(g) = f_1(gns) = \begin{cases} \chi(gns) & \rho_0(gns) \in \mathbf{B}(k_F) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \chi(gns) & \rho_0(g) \in \mathbf{B}(k_F)\rho_0(sn^{-1}) \\ 0 & else \end{cases} = f_{sn^{-1}}(g)$$

so that

$$nsf_1 = f_{sn^{-1}}$$

for all $n \in \mathbf{N}(k_F)$. In particular, $f_{sn} = n^{-1}s \cdot f_1 \in \mathcal{O}_C K_0 \cdot f_1$ for any basis element, establishing the proposition when E/F is unramified.

When E/F is ramified, we obtain similarly that $f_{sn} = n^{-1}s \cdot f_1 \in \mathcal{O}_C K_0 \cdot f_1$ for any $n \in \mathbf{N}'(k_F)$, finishing the proof. □

We next describe the intersection of L_0 with $V_{01} = \left(ind_B^G \chi\right)^{I(1)} = V_0^{I(1)}$.

Lemma 7.4. *A system of representatives for $I(1)/K_0(1)$ is $\mathbf{N}(k_F)$ when E/F is unramified, or by $\mathbf{N}'(k_F)$ when E/F is ramified. Thus, a basis of $L_0^{I(1)}$ is given by*

$$f_1, \sum_{n \in \mathbf{N}(k_F)} f_{sn}$$

in the former case, and by

$$f_1, \sum_{n \in \mathbf{N}'(k_F)} f_{sn}$$

in the latter.

Proof. Consider the natural reduction map $\rho : K_0 \to \mathbf{U}_3(k_F)$, when E/F is unramified, and $\rho : K_0 \to \mathbf{O}_3(k_F)$ when E/F is ramified.

Then $K_0(1) = \ker \rho$, showing that $I(1)/K_0(1) \simeq \rho(I(1))$. However, $I(1) = \rho^{-1}(\mathbf{N}(k_F))$ when E/F is unramified, while $I(1) = \rho^{-1}(\mathbf{N}'(k_F))$ when E/F is ramified.

This establishes the first claim.

It follows that in both cases

$$B \cdot I(1) = \bigcup_n BnK_0(1) = BK_0(1)$$

and

$$BsI(1) = \bigcup_n BsnK_0(1) = \coprod_n BsnK_0(1)$$

Let $\phi_1, \phi_s \in V_{01} = \left(ind_B^G \chi\right)^{I(1)}$ be the functions with supports $B \cdot I(1) = B \cdot K_0(1)$, $BsI(1)$ and value 1 at $1, s$ respectively.

As $I(1) = \coprod_n nK_0(1)$, and $\phi_s(sn) = \phi_s(s) = 1$, we see that r_0 with respect to the above bases is

$$r_0(\phi_1) = f_1, \quad r_0(\phi_s) = \sum_n f_{sn} \tag{7.1}$$

and the result follows. □

We also have a corresponding integral structure in $V_1 = \left(ind_B^G \chi\right)^{K_1(1)}$, which we now introduce.

7.1.2. The integral structure L_1

Let L_1 be the $\mathcal{O}_C$-integral structure of the C-representation of K_1 on $V_1 = \left(ind_{B \cap K_1}^{K_1} \eta\right)^{K_1(1)}$ given by the functions with values in $\mathcal{O}_C$.

We denote by $h_g \in L_1$ the function of support $(B \cap K_1)\, gK_1(1)$ and value 1 at g.

When E/F is unramified, a system of representatives of $B \cap K_1 \backslash K_1 / K_1(1) \simeq \mathbf{M}(k_F)\mathbf{Z}(k_F)\backslash \mathbf{H}(k_F)$ (see Lemma 6.19) is

$$1, tz \text{ for } z \in \mathbf{Z}(k_F), \quad t = \begin{pmatrix} 0 & 0 & \pi^{-1} \\ 0 & 1 & 0 \\ \overline{\pi} & 0 & 0 \end{pmatrix}$$

by the Bruhat decomposition $\mathbf{H}(k_F) = \mathbf{M}(k_F)\mathbf{Z}(k_F) \coprod \mathbf{M}(k_F)\mathbf{Z}(k_F)s\mathbf{Z}(k_F)$.

When E/F is ramified, a system of representatives of $B \cap K_1 \backslash K_1/K_1(1) \simeq \mathbf{M}'(k_F)\mathbf{Z}'(k_F)\backslash\mathbf{H}'(k_F)$ (see Lemma 6.19) is

$$\{1, tz\}_{z\in\mathbf{Z}'(k_F)}$$

Therefore, an $\mathcal{O}_C$-basis of L_1 is $\{h_1, h_{tz}\}_{z\in\mathbf{Z}(k_F)}$ if E/F is unramified, and $\{h_1, h_{tz}\}_{z\in\mathbf{Z}'(k_F)}$ if E/F is ramified.

Proposition 7.5. *The $\mathcal{O}_C K_1$-module L_1 is cyclic, generated by h_t, i.e. $L_1 = \mathcal{O}_C K_1 \cdot h_t$.*

Proof. For any $z \in \mathbb{F}_q^-$ and any $g \in K_1$, one has

$$n_{0,\pi^{-1}z}th_1(g) = h_1(gn_{0,\pi^{-1}z}t) = \begin{cases}\chi(gn_{0,\pi^{-1}z}t) & \rho_1(gn_{0,\pi^{-1}z}t) \in \mathbf{M}(k_F)\mathbf{Z}(k_F) \\ 0 & else\end{cases} =$$

$$= \begin{cases}\chi(gn_{0,\pi^{-1}z}t) & \rho_1(g) \in \mathbf{M}(k_F)\mathbf{Z}(k_F)\rho_1(tn_{0,\pi^{-1}z}) \\ 0 & else\end{cases} = h_{tn_{0,\pi^{-1}\bar{z}}}(g)$$

since $\mathbf{M}(k_F)\mathbf{Z}(k_F)\rho_1(tn_{0,\pi^{-1}z}) = \mathbf{M}(k_F)\mathbf{Z}(k_F)\rho_1(tn_{0,\pi^{-1}\bar{z}})$. Thus

$$n_{0,\pi^{-1}z}th_1 = h_{tn_{0,\pi^{-1}\bar{z}}}$$

for $z \in \mathbb{F}_q^-$. In particular, for all $z \in \mathbb{F}_q^-$, one has $h_{tn_{0,\pi^{-1}z}} = n_{0,\pi^{-1}\bar{z}}th_1 \in \mathcal{O}_C K_1 \cdot h_1$, showing the proposition when E/F is unramified.
Similarly, for any $z \in \mathbb{F}_q$ and any $g \in K_1$, one has

$$n_{0,\pi^{-1}z}th_1(g) = h_1(gn_{0,\pi^{-1}z}t) = \begin{cases}\chi(gn_{0,\pi^{-1}z}t) & \rho_1(gn_{0,\pi^{-1}z}t) \in \mathbf{M}'(k_F)\mathbf{Z}'(k_F) \\ 0 & else\end{cases} =$$

$$= \begin{cases}\chi(gn_{0,\pi^{-1}z}t) & \rho_1(g) \in \mathbf{M}'(k_F)\mathbf{Z}'(k_F)\rho_1(tn_{0,\pi^{-1}z}^{-1}) \\ 0 & else\end{cases} = h_{tn_{0,-\pi^{-1}z}}(g)$$

since $\mathbf{M}'(k_F)\mathbf{Z}'(k_F)\rho_1(tn_{0,\pi^{-1}z}^{-1}) = \mathbf{M}'(k_F)\mathbf{Z}'(k_F)\rho_1(tn_{0,\pi^{-1}\bar{z}})$. Thus

$$n_{0,\pi^{-1}z}th_1 = h_{tn_{0,-\pi^{-1}z}}$$

But $h_t = th_1$, hence $h_1 = th_t$, thus establishing the proposition. □

Lemma 7.6. *A system of representatives for $I(1)/K_1(1)$ is $(sn_{0,\pi z}s)_{z\in\mathbb{F}_q^-}$ when E/F is unramified, and $(sn_{0,\pi z}s)_{z\in\mathbb{F}_q}$ when E/F is ramified. Thus, when E/F is unramified*

$$r_1(\phi_1) = h_1 + \sum_{y\in\mathbb{F}_q^-} \eta^\star(y) \cdot h_{tn_{0,\pi^{-1}y}}, \quad r_1(\phi_s) = \lambda^{-1} \cdot h_t$$

and when E/F is ramified,

$$r_1(\phi_1) = h_1 + \sum_{y\in\mathbb{F}_q} \eta^{-1}(y) \cdot h_{tn_{0,\pi^{-1}y}}, \quad r_1(\phi_s) = \lambda^{-1} \cdot h_t$$

Proof. Note that when E/F is unramified, one has

$$I(1)/K_1(1) \simeq {}^{K_1}/{}_{I(1)} \backslash {}^{K_1}/{}_{K_1(1)} \simeq \mathbf{M}(k_F)\mathbf{Z}(k_F) \backslash \mathbf{H}(k_F)$$

so by the Bruhat decomposition $\mathbf{H}(k_F) = \mathbf{Z}_-(k_F)\mathbf{M}(k_F)\mathbf{Z}(k_F)$, we see that $(sn_{0,\pi z}s)_{z \in \mathbb{F}_q^-}$ is indeed a system of representatives for $I(1)/K_1(1)$.
When E/F is ramified, one has

$$I(1)/K_1(1) \cong {}^{K_1}/{}_{I(1)} \backslash {}^{K_1}/{}_{K_1(1)} \cong \mathbf{M}'(k_F)\mathbf{Z}'(k_F) \backslash \mathbf{H}'(k_F)$$

so by the Bruhat decomposition $\mathbf{H}'(k_F) = \mathbf{Z}'_-(k_F)\mathbf{M}'(k_F)\mathbf{Z}'(k_F)$, we see that $(sn_{0,\pi z}s)_{z \in \mathbb{F}_q}$ is indeed a system of representatives for $I(1)/K_1(1)$.
Further, we have for any $0 \neq z \in \mathbb{F}_q$, that

$$sn_{0,\pi z}s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \pi z & 0 & 1 \end{pmatrix} = \begin{pmatrix} -z^{-1} & 0 & \pi^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & z \end{pmatrix} \begin{pmatrix} 0 & 0 & \pi^{-1} \\ 0 & 1 & 0 \\ \pi & 0 & z^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} -z^{-1} & 0 & \pi^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & z \end{pmatrix} \begin{pmatrix} 0 & 0 & \pi^{-1} \\ 0 & 1 & 0 \\ \pi & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \pi^{-1}z^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in Btn_{0,\pi^{-1}z^{-1}}$$

It follows that when E/F is unramified, one has

$$B \cdot I(1) = \bigcup_{z \in \mathbb{F}_q^-} Bsn_{0,\pi z}sK_1(1) = PK_1(1) \cup \left(\bigcup_{0 \neq z \in \mathbb{F}_q^-} Btn_{0,\pi^{-1}z}K_1(1) \right)$$

and

$$BsI(1) = BsK_1(1) \cup \left(\bigcup_{0 \neq z \in \mathbb{F}_q^-} Bn_{0,\pi z}sK_1(1) \right) = BsK_1(1)$$

while if E/F is ramified, one has

$$B \cdot I(1) = \bigcup_{z \in \mathbb{F}_q} Bsn_{0,\pi z}sK_1(1) = BK_1(1) \cup \left(\bigcup_{0 \neq z \in \mathbb{F}_q} Btn_{0,\pi^{-1}z}K_1(1) \right)$$

and

$$BsI(1) = BsK_1(1) \cup \left(\bigcup_{0 \neq z \in \mathbb{F}_q} Bn_{0,\pi z}sK_1(1) \right) = BsK_1(1)$$

Moreover

$$\phi_1(tn_{0,\pi^{-1}z}) = \phi_1 \left(\begin{pmatrix} \overline{z}^{-1} & 0 & \pi^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & z \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \pi z^{-1} & 0 & 1 \end{pmatrix} \right) =$$

$$= \chi(\overline{z}^{-1}) = \begin{cases} \eta^\star(z) & E/F \quad unramified \\ \eta^{-1}(z) & E/F \quad ramified \end{cases}$$

here we use $z = -\overline{z}$.

It follows that, when E/F is unramified

$$r_1(\phi_1) = h_1 + \sum_{y \in \mathbb{F}_q^-} \eta^\star(y) \cdot h_{tn_{0,\pi^{-1}y}}$$

where we extend $\eta^\star$ to $\mathbb{F}_q^-$ such that $\eta^\star(0) = 0$. When E/F is ramified, we have

$$r_1(\phi_1) = h_1 + \sum_{y \in \mathbb{F}_q} \eta^{-1}(y) \cdot h_{tn_{0,\pi^{-1}y}}$$

It is convenient to write $t = sp = spss = p^{-1}s$ where $p = \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi^{-1} \end{pmatrix}$.

Then it is easy to see that

$$\phi_s(t) = \phi_s(p^{-1}s) = \chi(p^{-1}) \cdot \phi_s(s) = \chi(p^{-1}) = \lambda^{-1}$$

showing that $r_1(\phi_s) = \lambda^{-1} h_t$. $\square$

Now that we have explicit description of r_0, r_1 in terms of bases, we proceed with the construction of zig-zags, as in Corollary 6.18 to prove Theorem 7.1.

7.2. Proof of Main Theorem

7.2.1. Proof of the criterion

We will prove the theorem one step at a time. Begin with the following proposition, establishing necessity of one of the conditions.

Proposition 7.7. *Let $\chi : E^\times \to C$ be a tamely ramified character, such that $ind_B^G \chi$ is integral. Let $\lambda = \chi(\pi)$. Then $\lambda^{-1} \in \mathcal{O}_C$.*

Proof. We proceed with the notations from the previous section.

Consider $L_0 = \mathcal{O}_C f_1 + \sum_{n \in \mathbf{N}(k_F)} \mathcal{O}_C f_{sn}$ when E/F is unramified or $L_0 = \mathcal{O}_C f_1 + \sum_{n \in \mathbf{N}'(k_F)} \mathcal{O}_C f_{sn}$ when E/F is ramified. Then $r_0^{-1}(L_0) = \mathcal{O}_C \phi_1 \oplus \mathcal{O}_C \phi_s$, hence

$$L_1' = K_1 \cdot r_1\left(r_0^{-1}(L_0)\right) = \mathcal{O}_C K_1 \cdot r_1(\phi_1) + \lambda^{-1} \mathcal{O}_C K_1 h_t$$

By Proposition 7.5, we know that $L_1 = \mathcal{O}_C K_1 h_t$. Therefore, if E/F is unramified,

$$r_1(\phi_1) = h_1 + \sum_{0 \neq y \in \mathbb{F}_q^-} \eta^\star(y) h_{tn_{0,\pi^{-1}y}} \in L_1 = \mathcal{O}_C K_1 h_t$$

while if E/F is ramified

$$r_1(\phi_1) = h_1 + \sum_{0 \neq y \in \mathbb{F}_q} \eta^{-1}(y) h_{tn_{0,\pi^{-1}y}} \in L_1 = \mathcal{O}_C K_1 h_t$$

and it follows that $\lambda^{-1} r_1(\phi_1) \in L_1^{'}$, so that $\lambda^{-1} f_1 \in r_0(r_1^{-1}(L_1)) \subset z(L_0)$.
But $L_0 = \mathcal{O}_C K_0 f_1$, hence $\lambda^{-1} L_0 \subset z(L_0)$. If the sequence of zigzags $(z^n(L_0))_{n \geq 0}$ is finite, then we must have $\lambda^{-1} \in \mathcal{O}_C$.
By Corollary 6.18, if $\lambda^{-1} \notin \mathcal{O}_C$ then $ind_B^G \chi$ is not integral. □

This also yields the easiest case for sufficiency, namely

Corollary 7.8. *If $\lambda \in \mathcal{O}_C^{\times}$, then $ind_B^G \chi$ is integral.*

Proof. Consider the description of $L_1^{'}$ given above. If $\lambda \in \mathcal{O}_C^{\times}$, it follows that $\lambda^{-1} \mathcal{O}_C K_1 h_t = \mathcal{O}_C K_1 h_t = L_1$.
Since $r_1(\phi_1) \in L_1$, it follows that $L_1^{'} = L_1$. Thus we see that $r_0^{-1}(L_0) = r_1^{-1}(L_1)$, and by Corollary 6.16, we see that $ind_B^G \chi$ is integral. □

Next, we turn to compute an explicit basis for $L_1^{'}$.

Lemma 7.9. *For any $z \in \mathbb{F}_q^-$ when E/F is unramified, and for any $z \in \mathbb{F}_q$ when E/F is ramified, denote*

$$H_z = tn_{0,\pi^{-1}z} \cdot r_1(\phi_1) \in L_1^{'}$$

Then $L_1^{'}$ is spanned over $\mathcal{O}_C$ by

$$\{H_z\}_{z \in \mathbb{F}_q^-}, \quad r_1(\phi_1), \quad \left\{\lambda^{-1} h_{tn_{0,\pi^{-1}z}}\right\}_{z \in \mathbb{F}_q^-}, \quad \lambda^{-1} h_1$$

when E/F is unramified, and by

$$\{H_z\}_{z \in \mathbb{F}_q}, \quad r_1(\phi_1), \quad \left\{\lambda^{-1} h_{tn_{0,\pi^{-1}z}}\right\}_{z \in \mathbb{F}_q}, \quad \lambda^{-1} h_1$$

when E/F is ramified.

Proof. Note that using Lemma 6.19, one has

$$K_1/I \simeq {}^{K_1/K_1(1)}/_{I/K_1(1)} \simeq \begin{cases} {}^{\mathbf{H}(k_F)}/_{\mathbf{M}(k_F)\mathbf{Z}_-(k_F)} & E/F \quad unramified \\ {}^{\mathbf{H}'(k_F)}/_{\mathbf{M}'(k_F)\mathbf{Z}_-^{'}(k_F)} & E/F \quad ramified \end{cases}$$

where $\mathbf{Z}_-(k_F) = s\mathbf{Z}(k_F)s$ and $\mathbf{Z}_-^{'}(k_F) = s\mathbf{Z}'(k_F)s$. Hence, by the Bruhat decomoposition, a system of representatives for K_1/I is

$$\{1, tn_{0,\pi^{-1}z} \mid z \in \mathbb{F}_q^-\} \text{ if } E/F \text{ is unramified}$$

$$\{1, tn_{0,\pi^{-1}z} \mid z \in \mathbb{F}_q\} \text{ if } E/F \text{ is ramified}$$

Note that as $r_0^{-1}(L_0)$ is I-invariant, and r_1 is I-equivariant, it is enough to consider the action of these representatives in order to obtain an explicit description of $L_1^{'}$. That is

$$L_1^{'} = \mathcal{O}_C r_1(\phi_1) + \sum_{z \in \mathbb{F}_q^-} \mathcal{O}_C tn_{0,\pi^{-1}z} r_1(\phi_1) + \lambda^{-1} L_1$$

when E/F is unramified, and

$$L_1^{'} = \mathcal{O}_C r_1(\phi_1) + \sum_{z \in \mathbb{F}_q} \mathcal{O}_C tn_{0,\pi^{-1}z} r_1(\phi_1) + \lambda^{-1} L_1$$

when E/F is ramified, but this is just the required result. □

Lemma 7.10. *Recall that we have defined Corollary 6.18, $z(L_0) = K_0 \cdot r_0(r^{-1}(L_1))$. We now have, when E/F is unramified*

$$q^{1/2}\lambda \cdot r_0(\phi_s) \in z(L_0)$$

and when E/F is ramified

$$q\lambda \cdot r_0(\phi_s) \in z(L_0)$$

Proof. We first note that when E/F is unramified

$$H_z = tn_{0,\pi^{-1}z} \cdot \left(h_1 + \sum_{0 \neq y \in \mathbb{F}_q^-} \eta^\star(y) h_{tn_{0,\pi^{-1}y}} \right) =$$

$$= h_t + \eta^\star(z) h_1 + \sum_{0, z \neq y \in \mathbb{F}_q^-} \eta^\star(y) \cdot \eta(\overline{y-z}) \cdot h_{tn_{0,\pi^{-1}(y-z)^{-1}}} =$$

$$= \eta^\star(z) h_1 + h_t + \sum_{0, -z^{-1} \neq y \in \mathbb{F}_q^-} \eta^\star(y^{-1} + z) \cdot \eta^\star(y) \cdot h_{tn_{0,\pi^{-1}y}} =$$

$$= h_t + \eta^\star(z) h_1 + \sum_{0 \neq y \in \mathbb{F}_q^-} \eta^\star(1 + yz) \cdot h_{tn_{0,\pi^{-1}y}}$$

If E/F is ramified, we see that

$$H_z = tn_{0,\pi^{-1}z} \cdot \left(h_1 + \sum_{0 \neq y \in \mathbb{F}_q} \eta^{-1}(y) h_{tn_{0,\pi^{-1}y}} \right) =$$

$$= h_t + \eta^{-1}(z) h_1 + \sum_{0, z \neq y \in \mathbb{F}_q} \eta^{-1}(y) \cdot \eta(\overline{y-z}) \cdot h_{tn_{0,\pi^{-1}(y-z)^{-1}}} =$$

$$= \eta^{-1}(z)h_1 + h_t + \sum_{0,-z^{-1}\neq y\in\mathbb{F}_q} \eta^{-1}(y^{-1}+z)\cdot\eta^{-1}(y)\cdot h_{tn_{0,\pi^{-1}y}} =$$

$$= h_t + \eta^{-1}(z)h_1 + \sum_{0\neq y\in\mathbb{F}_q} \eta^{-1}(1+yz)\cdot h_{tn_{0,\pi^{-1}y}}$$

When E/F is unramified, consider the sum

$$\sum_{z\in\mathbb{F}_q^-} H_z = q^{1/2}h_t + \sum_{z\in\mathbb{F}_q^-} \eta^\star(z)\cdot h_1 + \sum_{0\neq y\in\mathbb{F}_q^-}\sum_{z\in\mathbb{F}_q^-} \eta^\star(1+yz)\cdot h_{tn_{0,\pi^{-1}y}}$$

and when E/F is ramified, consider the sum

$$\sum_{z\in\mathbb{F}_q} H_z = qh_t + \sum_{z\in\mathbb{F}_q} \eta^{-1}(z)\cdot h_1 + \sum_{0\neq y\in\mathbb{F}_q}\sum_{z\in\mathbb{F}_q} \eta^{-1}(1+yz)\cdot h_{tn_{0,\pi^{-1}y}}$$

Now, for any $0\neq y\in\mathbb{F}_q^-$ and for any $z\in\mathbb{F}_q^-$, as $y=-\overline{y}$ and $z=-\overline{z}$, we see that $\overline{1+yz}=1+yz$, hence $1+yz\in\mathbb{F}_{q^{1/2}}$.
Moreover, if $1+yz_1=1+yz_2$, then as $y\neq 0$, it follows that $z_1=z_2$, hence the map $z\mapsto 1+yz$ from $\mathbb{F}_q^-$ to $\mathbb{F}_{q^{1/2}}$ is injective, and as these are finite sets of the same size, bijective. The same holds for the map $z\mapsto 1+yz:\mathbb{F}_q\to\mathbb{F}_q$.
It follows that $\sum_{z\in\mathbb{F}_q^-}\eta^\star(1+yz)=\sum_{x\in\mathbb{F}_{q^{1/2}}}\eta^\star(x)$. Also, $\sum_{z\in\mathbb{F}_q}\eta^{-1}(1+yz)=\sum_{z\in\mathbb{F}_q}\eta^{-1}(x)$.
When E/F is ramified, it immediately follows that if $\eta\neq 1$

$$\sum_{z\in\mathbb{F}_q} H_z = qh_t$$

while if $\eta=1$, then

$$\sum_{z\in\mathbb{F}_q} H_z = qh_t + (q-1)h_1 + (q-1)\cdot\sum_{0\neq y\in\mathbb{F}_q} h_{tu_{0,\pi^{-1}y}} = qh_t + (q-1)\cdot r_1(\phi_1)$$

Therefore, in any case, $qh_t\in L_1^{'}$, hence

$$q\lambda\cdot\sum_{n\in\mathbf{N}'(k_F)} f_{sn} = r_0(q\lambda\phi_s) = r_0(r_1^{-1}(qh_t))\in r_0(r_1^{-1}(L_1))\subset z(L_0)$$

When E/F is unramified, by Lemma 2.24, we see that :
If $\eta\restriction_{\mathbb{F}^\times_{q^{1/2}}}\neq 1$, we have

$$\sum_{z\in\mathbb{F}_q^-} H_z = q^{1/2}h_t$$

If $\eta\restriction_{\mathbb{F}^\times_{q^{1/2}}}=1$, and $p\neq 2$, we have

$$\sum_{z\in\mathbb{F}_q^-} H_z = q^{1/2}h_t + (q^{1/2}-1)\eta^\star(i)h_1 + (q^{1/2}-1)\cdot\sum_{0\neq y\in\mathbb{F}_q^-} h_{tn_{0,\pi^{-1}y}}$$

and if $\eta\restriction_{\mathbb{F}^\times_{q^{1/2}}}=1$, and $p=2$, we have

$$\sum_{z\in\mathbb{F}_q^-} H_z = q^{1/2}h_t + (q^{1/2}-1)h_1 + (q^{1/2}-1)\cdot \sum_{0\neq y\in\mathbb{F}_q^-} h_{tn_{0,\pi^{-1}y}}$$

It follows that if $\eta\restriction_{\mathbb{F}^\times_{q^{1/2}}}=1$, and $p\neq 2$ then

$$\sum_{z\in\mathbb{F}_q^-} H_z = q^{1/2}h_t + (q^{1/2}-1)\eta^\star(i)\cdot\left(h_1 + \sum_{0\neq y\in\mathbb{F}_q^-}\eta^\star(i)h_{tn_{0,\pi^{-1}y}}\right) =$$

$$= q^{1/2}h_t+(q^{1/2}-1)\cdot\left(h_1+\sum_{0\neq y\in\mathbb{F}_q^-}\eta^\star(y)\cdot h_{tn_{0,\pi^{-1}y}}\right) = q^{1/2}h_t+(q^{1/2}-1)\cdot r_1(\phi_1)$$

while if $\eta\restriction_{\mathbb{F}^\times_{q^{1/2}}}=1$ and $p=2$, then

$$\sum_{z\in\mathbb{F}_q^-} H_z = q^{1/2}h_t + (q^{1/2}-1)\cdot\left(h_1 + \sum_{0\neq y\in\mathbb{F}_q^-}h_{tn_{0,\pi^{-1}y}}\right) =$$

$$= q^{1/2}h_t+(q^{1/2}-1)\cdot\left(h_1+\sum_{0\neq y\in\mathbb{F}_q^-}\eta^\star(y)\cdot h_{tn_{0,\pi^{-1}y}}\right) = q^{1/2}h_t+(q^{1/2}-1)\cdot r_1(\phi_1)$$

In any case, we see that $q^{1/2}h_t\in L_1^{'}$, hence

$$q^{1/2}\lambda\cdot\sum_{n\in\mathbf{N}(k_F)} f_{sn} = r_0(q^{1/2}\lambda\phi_s) = r_0(r_1^{-1}(q^{1/2}h_t))\in r_0(r_1^{-1}(L_1))\subset z(L_0)$$

□

Next, we compute the $\mathcal{O}_CK_0$-module M_0 generated by $r_0(\phi_s)$ (which is $\sum_{n\in\mathbf{N}(k_F)} f_{sn}$ if E/F is unramified, and $\sum_{n\in\mathbf{N}'(k_F)} f_{sn}$ if E/F is ramified).

Proposition 7.11. *Set $M_0=\mathcal{O}_CK_0\cdot r_0(\phi_s)$, and for any $n\in\mathbf{N}(k_F)$, denote, when E/F is unramified,*

$$F_n = ns\sum_{n'\in\mathbf{N}(k_F)} f_{sn'}$$

when E/F is ramified, denote for any $n\in\mathbf{N}'(k_F)$

$$F_n = ns\sum_{n'\in\mathbf{N}'(k_F)} f_{sn'}$$

Then M_0 is spanned over $\mathcal{O}_C$, when E/F is unramified by

$$r_0(\phi_s), \quad \{F_n\}_{n\in\mathbf{N}(k_F)}$$

and when E/F is ramified, by

$$r_0(\phi_s), \quad \{F_{n'}\}_{n\in\mathbf{N}'(k_F)}$$

Proof. Note that using Lemma 6.19, one has

$$K_0/Z(G)\cdot I(1) \simeq {}^{K_0/K_0(1)}/_{Z(G)\cdot I(1)/K_0(1)} \simeq \begin{cases} {}^{\mathbf{G}(k_F)}/_{Z(G)\cdot\mathbf{N}(k_F)} & E/F \quad unramified \\ {}^{\mathbf{O_3}(k_F)}/_{Z(O_3)\cdot\mathbf{N}'(k_F)} & E/F \quad ramified \end{cases}$$

By the Bruhat decomposition

$$\mathbf{G}(k_F) = \mathbf{B}(k_F)\coprod \mathbf{B}(k_F)s\mathbf{N}(k_F) = \mathbf{M}(k_F)\mathbf{N}(k_F)\coprod \mathbf{M}(k_F)\mathbf{N}(k_F)s\mathbf{N}(k_F)$$

$$\mathbf{O_3}(k_F) = \mathbf{B}'(k_F)\coprod \mathbf{B}'(k_F)s\mathbf{N}'(k_F) = \mathbf{M}'(k_F)\mathbf{N}'(k_F)\coprod \mathbf{M}'(k_F)\mathbf{N}'(k_F)s\mathbf{N}'(k_F)$$

It follows that a system of representatives for $K_0/Z(G)\cdot I(1)$ is

$$\left\{d_a, d_a ns \mid a\in\mathbb{F}_q^\times, \quad n\in\mathbf{N}(k_F)\right\}$$

when E/F is unramified, and

$$\left\{d_a^{'}, d_a^{'}ns \mid a\in\mathbb{F}_q^\times, \quad n\in\mathbf{N}'(k_F)\right\}, \quad d_a^{'} = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}$$

when E/F is ramified.

As for any $n_{c,y}\in\mathbf{N}(k_F)$ and any $a\in\mathbb{F}_q^\times$

$$\begin{aligned} sn_{c,y}d_a &= \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}\begin{pmatrix} 1&c&y\\0&1&-\overline{c}\\0&0&1 \end{pmatrix}\begin{pmatrix} a&0&0\\0&1&0\\0&0&\overline{a}^{-1} \end{pmatrix} = \\ &= \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}\begin{pmatrix} a&c&\overline{a}^{-1}y\\0&1&-\overline{a}^{-1}\overline{c}\\0&0&\overline{a}^{-1} \end{pmatrix} = \begin{pmatrix} 0&0&\overline{a}^{-1}\\0&1&-\overline{a}^{-1}\overline{c}\\a&c&\overline{a}^{-1}y \end{pmatrix} = \\ &= \begin{pmatrix} \overline{a}^{-1}&0&0\\0&1&0\\0&0&a \end{pmatrix}\begin{pmatrix} 0&0&1\\0&1&-\overline{a}^{-1}\overline{c}\\1&a^{-1}c&a^{-1}\overline{a}^{-1}y \end{pmatrix} = \\ &= \begin{pmatrix} \overline{a}^{-1}&0&0\\0&1&0\\0&0&a \end{pmatrix}\begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}\begin{pmatrix} 1&a^{-1}c&a^{-1}\overline{a}^{-1}y\\0&1&-\overline{a}^{-1}\overline{c}\\0&0&1 \end{pmatrix} = \\ &= d_{\overline{a}^{-1}}sn_{a^{-1}c,a^{-1}\overline{a}^{-1}y} \end{aligned}$$

we thus have for any $g \in K_0$, when E/F is unramified,

$$d_a f_1(g) = \begin{cases} \chi(g d_a) & \rho_0(g) \in \mathbf{B}(k_F)\rho_0(d_{a^{-1}}) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \chi(g) \cdot \eta(a) & \rho_0(g) \in \mathbf{B}(k_F) \\ 0 & else \end{cases} = \eta(a) f_1(g)$$

$$d_a f_{sn_{c,y}}(g) = \begin{cases} \chi(g d_a n_{c,y}^{-1} s) = \chi(g n_{ac,a\overline{a}y}^{-1} s d_{\overline{a}^{-1}}) & \rho_0(g) \in \mathbf{B}(k_F)\rho_0(sn_{c,y} d_{a^{-1}}) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \eta^\star(a) \cdot \chi(g n_{ac,a\overline{a}y}^{-1} s) & \rho_0(g) \in \mathbf{B}(k_F)\rho_0(sn_{ac,a\overline{a}y}) \\ 0 & else \end{cases} = \eta^\star(a) f_{sn_{ac,a\overline{a}y}}(g)$$

showing that

$$d_a f_1 = \eta(a) \cdot f_1, \quad d_a f_{sn_{c,y}} = \eta^\star(a) \cdot f_{sn_{ac,a\overline{a}y}} \tag{7.2}$$

Similarly, as for any $c \in \mathbb{F}_q$ and any $a \in \mathbb{F}_q^\times$

$$\begin{aligned}
sn_c d_a^{'} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & c & -\frac{c^2}{2} \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} = \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & c & -a^{-1}\frac{c^2}{2} \\ 0 & 1 & -a^{-1}c \\ 0 & 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^{-1} \\ 0 & 1 & -a^{-1}c \\ a & c & -a^{-1}\frac{c^2}{2} \end{pmatrix} = \\
&= \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -a^{-1}c \\ 1 & a^{-1}c & -a^{-2}\frac{c^2}{2} \end{pmatrix} = \\
&= \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}c & -\frac{a^{-2}c^2}{2} \\ 0 & 1 & -a^{-1}c \\ 0 & 0 & 1 \end{pmatrix} = \\
&= d_{a^{-1}}^{'} sn_{a^{-1}c}
\end{aligned}$$

we have for any $g \in K_0$, when E/F is ramified and $p \neq 2$,

$$d_a^{'} f_1(g) = \begin{cases} \chi(g d_a^{'}) & \rho_0(g) \in \mathbf{B}'(k_F)\rho_0(d_{a^{-1}}^{'}) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \chi(g) \cdot \eta(a) & \rho_0(g) \in \mathbf{B}'(k_F) \\ 0 & else \end{cases} = \eta(a) f_1(g)$$

$$d_a^{'} f_{sn_c}(g) = \begin{cases} \chi(g d_a^{'} n_c^{-1} s) = \chi(g n_{ac}^{-1} s d_{a^{-1}}^{'}) & \rho_0(g) \in \mathbf{B}'(k_F)\rho_0(sn_c d_{a^{-1}}^{'}) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \eta^{-1}(a) \cdot \chi(g n_{ac}^{-1} s) & \rho_0(g) \in \mathbf{B'}(k_F)\rho_0(sn_{ac}) \\ 0 & else \end{cases} = \eta^{-1}(a) f_{sn_{ac}}(g)$$

showing that

$$d_a^{'} f_1 = \eta(a) \cdot f_1, \quad d_a^{'} f_{sn_c} = \eta^{-1}(a) \cdot f_{sn_{ac}} \tag{7.3}$$

When $p = 2$ and E/F is ramified, we see that

$$\begin{aligned} sn_{0,y} d_a^{'} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} = \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & a^{-1}y \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^{-1} \\ 0 & 1 & 0 \\ a & 0 & a^{-1}y \end{pmatrix} = \\ &= \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & a^{-2}y \end{pmatrix} = \\ &= \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & a^{-2}y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \\ &= d_{a^{-1}}^{'} sn_{0,a^{-2}y} \end{aligned}$$

so for any $g \in K_0$

$$d_a^{'} f_{sn_{0,y}}(g) = \begin{cases} \chi(g d_a^{'} n_{0,y}^{-1} s) = \chi(g n_{0,a^2 y}^{-1} s d_{a^{-1}}^{'}) & \rho_0(g) \in \mathbf{B'}(k_F)\rho_0(sn_{0,y} d_{a^{-1}}^{'}) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \eta^{-1}(a) \cdot \chi(g n_{0,a^2 y}^{-1} s) & \rho_0(g) \in \mathbf{B'}(k_F)\rho_0(sn_{0,a^2 y}) \\ 0 & else \end{cases} = \eta^{-1}(a) f_{sn_{0,a^2 y}}(g)$$

showing that $d_a^{'} f_{sn_{0,y}} = \eta^{-1}(a) \cdot f_{sn_{0,a^2 y}}$.

As $n_{c,y} \mapsto n_{ac,a\overline{a}y}$ is bijective on $\mathbf{N}(k_F)$, we see that

$$\mathcal{O}_C d_a \sum_{n \in \mathbf{N}(k_F)} f_{sn} = \mathcal{O}_C \eta^{\star}(a) \sum_{n \in \mathbf{N}(k_F)} f_{sn}$$

As $\eta(a)$ is a unit (note that η is a character of a finite group), we have

$$\mathcal{O}_C d_a \sum_{n \in \mathbf{N}(k_F)} f_{sn} = \mathcal{O}_C \sum_{n \in \mathbf{N}(k_F)} f_{sn}$$

Similarly, when E/F is ramified, we see that $c \mapsto ac$ and $y \mapsto a^2 y$ are bijective on $\mathbb{F}_q$, hence

$$\mathcal{O}_C d_a^{'} \sum_{c \in \mathbb{F}_q} f_{sn_c} = \mathcal{O}_C \eta^{-1}(a) \sum_{c \in \mathbb{F}_q} f_{sn_c} = \mathcal{O}_C \sum_{c \in \mathbb{F}_q} f_{sn_c}$$

and when $p = 2$

$$\mathcal{O}_C d_a^{'} \sum_{y \in \mathbb{F}_q} f_{sn_{0,y}} = \mathcal{O}_C \eta^{-1}(a) \sum_{y \in \mathbb{F}_q} f_{sn_{0,y}} = \mathcal{O}_C \sum_{y \in \mathbb{F}_q} f_{sn_{0,y}}$$

Next, note that, for E/F unramified,

$$sn_{c,y}s = \begin{pmatrix} \overline{y}^{-1} & -y^{-1}c & 1 \\ 0 & -y^{-1}\overline{y} & -\overline{c} \\ 0 & 0 & y \end{pmatrix} sn_{y^{-1}c,y^{-1}}$$

if $(c,y) \neq (0,0)$ (note that $y = 0 \Rightarrow c = 0$), and forE/F ramified, if $p \neq 2$,

$$sn_c s = \begin{pmatrix} -\frac{2}{c^2} & \frac{2}{c} & 1 \\ 0 & -1 & -c \\ 0 & 0 & -\frac{c^2}{2} \end{pmatrix} sn_{-\frac{2}{c}}$$

if $c \neq 0$. If E/F is ramified, and $p = 2$, we see that

$$sn_{0,y}s = \begin{pmatrix} y^{-1} & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & y \end{pmatrix} sn_{0,y^{-1}}$$

if $y \neq 0$.

Hence we have for any $g \in K_0$, for E/F unramified,

$$n_{b,z}sf_s(g) = f_s(gn_{b,z}s) = \begin{cases} \chi(gn_{b,z}) & \rho_0(gn_{b,z}s) \in \mathbf{B}(k_F)s \\ 0 & else \end{cases} =$$

$$= \begin{cases} \chi(g) & \rho_0(g) \in \mathbf{B}(k_F) \\ 0 & else \end{cases} = f_1(g)$$

while for E/F ramified, $p \neq 2$,

$$n_b s f_s(g) = f_s(gn_b s) = \begin{cases} \chi(gn_b) & \rho_0(gn_b s) \in \mathbf{B}'(k_F)s \\ 0 & else \end{cases} =$$

$$= \begin{cases} \chi(g) & \rho_0(g) \in \mathbf{B}'(k_F) \\ 0 & else \end{cases} = f_1(g)$$

and for E/F ramified, $p = 2$,

$$n_{0,z}sf_s(g) = f_s(gn_{0,z}s) = \begin{cases} \chi(gn_{0,z}) & \rho_0(gn_{0,z}s) \in \mathbf{B}'(k_F)s \\ 0 & else \end{cases} =$$

$$= \begin{cases} \chi(g) & \rho_0(g) \in \mathbf{B}'(k_F) \\ 0 & else \end{cases} = f_1(g)$$

Further, for any $(c, y) \neq (0,0)$, when E/F is unramified,

$$n_{b,z}sf_{sn_{c,y}}(g) = f_{sn_{c,y}}(gn_{b,z}s) = \begin{cases} \chi(gn_{b,z}sn_{c,y}^{-1}s) & \rho_0(gn_{b,z}s) \in \mathbf{B}(k_F)\rho_0(sn_{c,y}) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \chi(gn_{b,z}sn_{c,y}^{-1}s) & \rho_0(g) \in \mathbf{B}(k_F)\rho_0(sn_{c,y}sn_{b,z}^{-1}) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \chi\left(gn_\star^{-1}s\begin{pmatrix} \overline{y}^{-1} & -y^{-1}c & 1 \\ 0 & -y^{-1}\overline{y} & -\overline{c} \\ 0 & 0 & y \end{pmatrix}^{-1}\right) & \rho_0(g) \in \mathbf{B}(k_F)\rho_0(sn_\star) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \eta(\overline{y}) \cdot \chi\left(gn_\star^{-1}s\right) & \rho_0(g) \in \mathbf{B}(k_F)\rho_0(sn_\star) \\ 0 & else \end{cases} = \eta(\overline{y}) \cdot f_{sn_\star}(g)$$

where $n_\star = n_{y^{-1}c-b,y^{-1}+\overline{z}+y^{-1}\overline{b}c}$
Similarly, for $c \neq 0$, when E/F is ramified, and $p \neq 2$,

$$n_bsf_{sn_c}(g) = f_{sn_c}(gn_bs) = \begin{cases} \chi(gn_bsn_c^{-1}s) & \rho_0(gn_bs) \in \mathbf{B}'(k_F)\rho_0(sn_c) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \chi(gn_bsn_c^{-1}s) & \rho_0(g) \in \mathbf{B}'(k_F)\rho_0(sn_csn_b^{-1}) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \chi\left(gn_{-\frac{2}{c}-b}^{-1}s\begin{pmatrix} -\frac{2}{c^2} & \frac{2}{c} & 1 \\ 0 & -1 & -c \\ 0 & 0 & -\frac{c^2}{2} \end{pmatrix}^{-1}\right) & \rho_0(g) \in \mathbf{B}'(k_F)\rho_0(sn_{-2c^{-1}-b}) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \eta\left(-\frac{c^2}{2}\right) \cdot \chi\left(gn_{-2c^{-1}-b}^{-1}s\right) & \rho_0(g) \in \mathbf{B}'(k_F)\rho_0(sn_{-2c^{-1}-b}) \\ 0 & else \end{cases} = \eta\left(-\frac{c^2}{2}\right) \cdot f_{sn_{-2c^{-1}-b}}(g)$$

and for $y \neq 0$, when E/F is ramified and $p = 2$,

$$n_{0,z}sf_{sn_{0,y}}(g) = f_{sn_{0,y}}(gn_{0,z}s) = \begin{cases} \chi(gn_{0,z}sn_{0,y}^{-1}s) & \rho_0(gn_{0,z}s) \in \mathbf{B}'(k_F)\rho_0(sn_{0,y}) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \chi(gn_{0,z}sn_{0,y}^{-1}s) & \rho_0(g) \in \mathbf{B}'(k_F)\rho_0(sn_{0,y}sn_{0,z}^{-1}) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \chi \left(g n_{0,y^{-1}+z}^{-1} s \begin{pmatrix} y^{-1} & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & y \end{pmatrix}^{-1} \right) & \rho_0(g) \in \mathbf{B}'(k_F)\rho_0(sn_{0,y^{-1}+z}) \\ 0 & else \end{cases} =$$

$$= \begin{cases} \eta(y) \cdot \chi \left(g n_{0,y^{-1}+z}^{-1} s \right) & \rho_0(g) \in \mathbf{B}'(k_F) sn_{0,y^{-1}+z} \\ 0 & else \end{cases} = \eta(y) \cdot f_{sn_{0,y^{-1}+z}}(g)$$

These show that for any $n_{b,z} \in \mathbf{N}(k_F)$, when E/F is unramified,

$$n_{b,z} s f_s = f_1, \quad n_{b,z} s f_{sn_{c,y}} = \eta(\overline{y}) \cdot f_{sn_{y^{-1}c-b,y^{-1}+\overline{z}+y^{-1}\overline{b}c}} \tag{7.4}$$

and for any $b \in \mathbb{F}_q$, when E/F is ramified, $p \neq 2$,

$$n_b s f_s = f_1, \quad n_b s f_{sn_c} = \eta\left(-\frac{c^2}{2}\right) \cdot f_{sn_{-2c^{-1}-b}} \tag{7.5}$$

while for any $z \in \mathbb{F}_q$, when E/F is ramified, and $p = 2$,

$$n_{0,z} s f_s = f_1, \quad n_{0,z} s f_{sn_{0,y}} = \eta(y) \cdot f_{sn_{0,y^{-1}+z}} \tag{7.6}$$

Therefore, for E/F unramified, we have

$$\begin{aligned} F_{b,z} = F_{n_{b,z}} \quad &:= \quad n_{b,z} s \sum_{n_{c,y} \in \mathbf{N}(k_F)} f_{sn_{c,y}} = f_1 + \sum_{0 \neq n_{c,y} \in \mathbf{N}(k_F)} \eta(\overline{y}) \cdot f_{sn_\star} = \\ &= \quad f_1 + \sum_{n_{c,y} \in \mathbf{N}(k_F)} \eta^\star(y + z - \overline{b}c) f_{sn_{c,y}} \end{aligned}$$

where the character $\eta^\star$ of $\mathbb{F}_q^\star$ is extended to a function on $\mathbb{F}_q$ vanishing on 0. For E/F ramified, when $p \neq 2$, we have

$$F_b := F_{n_b} = n_b s \sum_{c \in \mathbb{F}_q} f_{sn_c} = f_1 + \sum_{0 \neq c \in \mathbb{F}_q} \eta\left(-\frac{c^2}{2}\right) \cdot f_{sn_{-2c^{-1}-b}} =$$

$$= f_1 + \sum_{c \in \mathbb{F}_q} \eta^{-1}\left(-\frac{(b+c)^2}{2}\right) f_{sn_c}$$

and for E/F ramified with $p = 2$, we have

$$\begin{aligned} F_z := F_{n_{0,z}} \quad &= \quad n_{0,z} s \sum_{y \in \mathbb{F}_q} f_{sn_{0,y}} = f_1 + \sum_{0 \neq y \in \mathbb{F}_q} \eta(y) \cdot f_{sn_{0,y^{-1}+z}} = \\ &= \quad f_1 + \sum_{y \in \mathbb{F}_q} \eta^{-1}(y+z) f_{sn_{0,y}} \end{aligned}$$

We have, combining with (7.2), when E/F is unramified

$$d_aF_{b,z} = d_an_{b.z}s \sum_{n_{c,y}\in\mathbf{N}(k_F)} f_{sn_{c,y}} = \eta(a)f_1 + \eta^{\star}(a) \sum_{n_{c,y}\in\mathbf{N}(k_F)} \eta^{\star}(y+z-\bar{b}c)f_{sn_{ac,a\bar{a}y}} =$$

$$= \eta(a)\left(f_1 + \sum_{n_{c,y}\in\mathbf{N}(k_F)} \eta^{\star}(a\bar{a}y + a\bar{a}z - a\bar{a}\bar{b}c)f_{sn_{ac,a\bar{a}y}}\right) =$$

$$= \eta(a)\left(f_1 + \sum_{n_{c,y}\in\mathbf{N}(k_F)} \eta^{\star}(y + a\bar{a}z - \overline{(ab)}c)f_{sn_{c,y}}\right) = \eta(a)\cdot F_{ab,a\bar{a}z}$$

As $\eta(a)$ is a unit, we have

$$\mathcal{O}_C d_a n_{b,z} s \sum_{n_{c,y}\in\mathbf{N}(k_F)} f_{sn_{c,y}} = \mathcal{O}_C F_{ab,a\bar{a}z}$$

Similarly, when E/F is ramified and $p \neq 2$, we have

$$d'_a F_b = d'_a n_b s \sum_{c\in\mathbb{F}_q} f_{sn_c} = \eta(a)f_1 + \eta^{-1}(a)\sum_{c\in\mathbb{F}_q} \eta^{-1}\left(-\frac{(b+c)^2}{2}\right) f_{sn_{ac}} =$$

$$= \eta(a)\left(f_1 + \sum_{c\in\mathbb{F}_q} \eta^{-1}\left(-\frac{a^2(b+c)^2}{2}\right) f_{sn_{ac}}\right) =$$

$$= \eta(a)\left(f_1 + \sum_{c\in\mathbb{F}_q} \eta^{-1}\left(-\frac{(ab+c)^2}{2}\right) f_{sn_c}\right) = \eta(a)\cdot F_{ab}$$

As $\eta(a)$ is a unit, we have

$$\mathcal{O}_C d'_a n_b s \sum_{c\in\mathbb{F}_q} f_{sn_c} = \mathcal{O}_C F_{ab}$$

and when E/F is ramified with $p = 2$, we have

$$d'_a F_z = d'_a n_{0,z} s \sum_{y\in\mathbb{F}_q} f_{sn_{0,y}} = \eta(a)f_1 + \eta^{-1}(a)\sum_{y\in\mathbb{F}_q} \eta^{-1}(y+z) f_{sn_{0,a^2y}} =$$

$$= \eta(a)\left(f_1 + \sum_{y\in\mathbb{F}_q} \eta^{-1}\left(a^2(y+z)\right) f_{sn_{0,a^2y}}\right) =$$

$$= \eta(a)\left(f_1 + \sum_{y\in\mathbb{F}_q} \eta^{-1}(y + a^2z) f_{sn_{0,y}}\right) = \eta(a)\cdot F_{a^2z}$$

As $\eta(a)$ is a unit, we have

$$\mathcal{O}_C d_a^{'} n_{0,z} s \sum_{y \in \mathbb{F}_q} f_{sn_{0,y}} = \mathcal{O}_C F_{a^2 z}$$

We deduce that M_0 is the $\mathcal{O}_C$-module generated, when E/F is unramified, by

$$\sum_{n \in \mathbf{N}(k_F)} f_{sn}, (F_n)_{n \in \mathbf{N}(k_F)}$$

when E/F is ramified, and $p \neq 2$, by

$$\sum_{c \in \mathbb{F}_q} f_{sn_c}, (F_b)_{b \in \mathbb{F}_q}$$

and when E/F is ramified and $p = 2$, by

$$\sum_{y \in \mathbb{F}_q} f_{sn_{0,y}}, (F_z)_{z \in \mathbb{F}_q}$$

which is the statement. $\square$

With this in hand, we may complete the proof of the necessity of the criterion.

Proposition 7.12. *Let $\chi : E^\times \to C$ be a tamely ramified character, such that $ind_B^G \chi$ is integral. Let $\lambda = \chi(\pi)$. Then $\lambda q^2 \in \mathcal{O}_C$.*

Proof. Consider the sum, when E/F is unramified,

$$\sum_{n \in \mathbf{N}(k_F)} F_n = q^{3/2} f_1 + \sum_{n_{c,y} \in \mathbf{N}(k_F)} \left(\sum_{n_{b,z} \in \mathbf{N}(k_F)} \eta^\star(y + z - \bar{b}c) \right) f_{sn_{c,y}}$$

We also note that $n_{c,y} \cdot n_{b,z} = n_{y+z, y+z-b\bar{c}}$, hence we see that

$$\sum_{n_{b,z} \in \mathbf{N}(k_F)} \eta^\star(y + z - \bar{b}c) = \sum_{n_{b,z} \in \mathbf{N}(k_F)} \eta^\star(z)$$

By Lemma 2.24, we see that:
If η is trivial, (the unramified case)

$$\sum_{n_{b,z} \in \mathbf{N}(k_F)} F_{b,z} = q^{3/2} f_1 + (q^{3/2} - 1) \sum_{n_{c,y} \in \mathbf{N}(k_F)} f_{sn_{c,y}}$$

If $\eta \restriction_{\mathbb{F}_{q^{1/2}}^\times} \neq 1$, then

$$\sum_{n_{b,z} \in \mathbf{N}(k_F)} F_{b,z} = q^{3/2} f_1$$

If $\eta\restriction_{\mathbb{F}_{q^{1/2}}}= 1$ and $p \neq 2$, then for any $i \in \mathbb{F}_q^-$

$$\sum_{n_{b,z}\in\mathbf{N}(k_F)} F_{b,z} = q^{3/2} f_1 - q^{1/2}(q^{1/2}-1)\eta(i)\cdot \sum_{n_{c,y}\in\mathbf{N}(k_F)} f_{sn_{c,y}}$$

If $\eta\restriction_{\mathbb{F}_{q^{1/2}}}= 1$ and $p = 2$, then

$$\sum_{n_{b,z}\in\mathbf{N}(k_F)} F_{b,z} = q^{3/2} f_1 - q^{1/2}(q^{1/2}-1)\cdot \sum_{n_{c,y}\in\mathbf{N}(k_F)} f_{sn_{c,y}}$$

In any case, we see that $q^{3/2}f_1 \in M_0$. Being K_0-stable, by Proposition 7.3, M_0 contains $q^{3/2}L_0$.

By Lemma 7.10, the zigzag $z(L_0) = K_0 \cdot r_0(r_1^{-1}(L_1))$ contains $q^{1/2}\lambda M_0$, hence $q^2\lambda L_0$.

When E/F is ramified, we consider the sum

$$\sum_{b\in\mathbb{F}_q} F_b = qf_1 + \sum_{c\in\mathbb{F}_q}\left(\sum_{b\in\mathbb{F}_q}\eta^{-1}\left(-\frac{(b+c)^2}{2}\right)\right) f_{sn_c}$$

when $p \neq 2$ and the sum

$$\sum_{z\in\mathbb{F}_q} F_z = qf_1 + \sum_{y\in\mathbb{F}_q}\left(\sum_{z\in\mathbb{F}_q}\eta^{-1}(y+z)\right) f_{sn_{0,y}}$$

when $p = 2$.

Note that $n_b \cdot n_c = n_{b+c}$ and $n_{0,y}\cdot n_{0,z} = n_{0,y+z}$. As $b \mapsto n_b, z \mapsto n_{0,z} : k_E \to \mathbf{N}'(k_F)$ are bijective, we see that

$$\sum_{b\in\mathbb{F}_q}\eta^{-1}\left(-\frac{(b+c)^2}{2}\right) = \sum_{b\in\mathbb{F}_q}\eta^{-1}\left(-\frac{b^2}{2}\right)$$

Also

$$\sum_{z\in\mathbb{F}_q}\eta^{-1}(y+z) = \sum_{z\in\mathbb{F}_q}\eta^{-1}(z)$$

By Lemma 2.24, we see that:

If η is either ε_q or trivial

$$\sum_{b\in\mathbb{F}_q} F_b = qf_1 + \eta\left(-\frac{1}{2}\right)\cdot(q-1)\cdot\sum_{c\in\mathbb{F}_q} f_{sn_c}$$

else

$$\sum_{b\in\mathbb{F}_q} F_b = qf_1$$

Further, when $p = 2$, we have if $\eta = 1$

$$\sum_{z \in \mathbb{F}_q} F_z = qf_1 + (q-1) \cdot \sum_{y \in \mathbb{F}_q} f_{sn_{0,y}}$$

else

$$\sum_{z \in \mathbb{F}_q} F_z = qf_1$$

In any case, we see that $qf_1 \in M_0$. Being K_0-stable, by Proposition7.3, M_0 contains qL_0.

By Lemma 7.10, the zigzag $z(L_0) = K_0 \cdot r_0(r_1^{-1}(L_1))$ contains $q\lambda M_0$, hence $q^2\lambda L_0$.

Thus in all cases, $z(L_0)$ contains $q^2\lambda L_0$.

If the sequence of zigzags $(z^n(L_0))_{n\geq 0}$ is finite, then $q^2\lambda \in \mathcal{O}_C$. By Corollary 6.18, if $q^2\lambda \notin \mathcal{O}_C$, then $ind_B^G\chi$ is not integral. □

Corollary 7.13. *This shows that for $ind_P^G\chi$ to be integral, we must have $1 \leq |\lambda| \leq |q^{-2}|$.*

Remark 7.14. Note that by Corollary 6.21, as the contragredient representation of $ind_B^G\chi$ is $ind_B^G\chi^{-1}\omega^2$, and $\chi^{-1}\omega^2(\pi) = \lambda^{-1}q^{-2}$, we see that if $ind_B^G\chi$ is integral, so is $ind_B^G\chi^{-1}\omega^2$, showing that $1 \leq \left|\lambda^{-1}q^{-2}\right| \leq |q^{-2}|$, hence $1 \leq |\lambda| \leq |q^{-2}|$. This shows that our condition is compatible with Corollary 6.21.

It is also compatible with the isomorphism $ind_B^G\chi \simeq ind_B^G\chi^{\star}\omega^2$ by the same computation.

7.3. Proof of sufficiency

We have established the necessity of the condition, and turn now to sufficiency.

We assume, then, that $\lambda^{-1} \in \mathcal{O}_C$ and $q^2\lambda \in \mathcal{O}_C$. Further, by Corollary 7.8, we may assume $\lambda \notin \mathcal{O}_C$.

To go further, we need a Lemma.

Definition 7.15. For a function $a : \mathbb{F}_q \to \mathcal{O}_C$ and a character $\eta : \mathbb{F}_q^\times \to \mathcal{O}_C^\times$, we consider the *convolution of a with η,* denoted by $a * \eta$, and defined by

$$(a * \eta)(y) := \sum_{z \in \mathbb{F}_q} a(-z)\eta(y+z)$$

for all $y \in \mathbb{F}_q$, where we set $\eta(0) := 0$.

Definition 7.16. We say that $a * \eta$ is *constant modulo* $\lambda^{-1}\mathcal{O}_C$ if there exists some $c \in C$ such that $(a * \eta)(y) - c \in \lambda^{-1}\mathcal{O}_C$ for all $y \in \mathbb{F}_q$.

Lemma 7.17. $\sum_{z \in \mathbb{F}_q} a(z) \in q \cdot \mathcal{O}_C + \lambda^{-1} \cdot \mathcal{O}_C$ *if $a * \eta$ is constant modulo $\lambda^{-1}\mathcal{O}_C$.*

Proof. When the character η is trivial, the function $a * \eta + a = \sum_{z \in \mathbb{F}_q} a(z)$ is constant. If $a * \eta$ is constant modulo $\lambda^{-1}\mathcal{O}_C$, so is a, and

$$\sum_{z \in \mathbb{F}_q} a(z) \in q \cdot a(0) + \lambda^{-1}\mathcal{O}_C \subset q \cdot \mathcal{O}_C + \lambda^{-1} \cdot \mathcal{O}_C$$

When the character η is not trivial, we use the Fourier transform. We replace C by a finite extension in order to find a non-trivial character $\psi : \mathbb{F}_q \to \mathcal{O}_C$ to define the Fourier transform

$$\hat{f}(y) = \sum_{z \in \mathbb{F}_q} \psi(zy) f(z)$$

of a function $f : \mathbb{F}_q \to C$.

We denote by $\mathcal{R}$ the space of integral functions $f : \mathbb{F}_q \to \mathcal{O}_C$, by $\hat{\mathcal{R}}$ the image of $\mathcal{R}$ by Fourier transform, by $\delta_0 \in \mathcal{R}$ the characteristic function of 0, and by $\Delta \in \mathcal{R}$ the constant function $\Delta(y) = 1$.

The properties of the Fourier transform yield

$$\hat{\hat{f}} = qf, \hat{\Delta} = q\delta_0, \hat{\delta_0} = \Delta, \hat{\eta}(0) = 0$$

$\hat{\eta}(x)$ is a Gauss sum, and $\hat{\eta}(x)\widehat{\eta^{-1}}(x) = q\eta(-1)$ if $x \in \mathbb{F}_q^\times$. The Fourier transform of a convolution product $f * g$ is the product of the Fourier transforms, i.e.

$$\widehat{f * g} = \hat{f} \cdot \hat{g}$$

The lemma then states that $\hat{a}(0) \in (q + \lambda^{-1})\mathcal{O}_C$ for all $a \in \mathcal{R}$ such that $a * \eta \in \mathcal{O}_C\Delta + \lambda^{-1}\mathcal{R}$. By Fourier transform $a * \eta \in \mathcal{O}_C\Delta + \lambda^{-1}\mathcal{R}$ is equivalent to $\hat{a} \cdot \hat{\eta} \in \mathcal{O}_C q\delta_0 + \lambda^{-1}\hat{\mathcal{R}}$. Multiplying by $\widehat{\eta^{-1}}$, which vanishes only at zero, yields for nonzero elements

$$q\hat{a} = q\hat{a}(0)\delta_0 + \lambda^{-1}\widehat{\eta^{-1}} \cdot \hat{\phi}$$

for some $\phi \in \mathcal{R}$. The function $b = qa$ belongs to $q\mathcal{R}$. We have $\hat{b} = \hat{b}(0)\delta_0 + \lambda^{-1}\widehat{\eta^{-1}} \cdot \hat{\phi}$ and by Fourier transform $b = \beta\Delta + \lambda^{-1} \cdot \eta^{-1} * \phi$ where $b(0) = \beta + \lambda^{-1} \cdot (\eta^{-1} * \phi)(0)$, hence $\beta \in (q + \lambda^{-1}) \cdot \mathcal{O}_C$. But $\hat{a}(0) = \beta$, hence the result. □

We return to the proof of Theorem 7.1. By Lemma 7.9, the $\mathcal{O}_C$-module $L_1^{'} = K_1 \cdot r_1(r_0^{-1}(L_0)) = K_1 \cdot (\mathcal{O}_C r_1(\phi_1) + \mathcal{O}_C r_1(\phi_s))$ is spanned over $\mathcal{O}_C$ by

$$\{H_z\}_{z \in \mathbb{F}_q^-}, r_1(\phi_1), \{\lambda^{-1} h_{tn_{0,\pi^{-1}z}}\}_{z \in \mathbb{F}_q^-}, \lambda^{-1} h_1$$

if E/F is unramified, or by

$$\{H_z\}_{z \in \mathbb{F}_q}, r_1(\phi_1), \{\lambda^{-1} h_{tn_{0,\pi^{-1}z}}\}_{z \in \mathbb{F}_q}, \lambda^{-1} h_1$$

if E/F is ramified.

The $\mathcal{O}_C$-module $r_1^{-1}(L_1^{'}) = (L_1^{'})^{I(1)}$ is spanned over $\mathcal{O}_C$ by $(\lambda^{-1}L_1)^{I(1)}$, ϕ_1, and, when E/F is unramified, by linear combinations (we let $i \in \mathbb{F}_q^-$ be arbitrary)

$$\sum_{z\in\mathbb{F}_q^-} a(-z/i)H_z = \left(\sum_{z\in\mathbb{F}_{q^{1/2}}} a(-z)\right)\cdot h_t + \left(\sum_{z\in\mathbb{F}_{q^{1/2}}} a(-z)\eta^\star(zi)\right)\cdot h_1 + + \sum_{0\neq y\in\mathbb{F}_q^-}\left(\sum_{z\in\mathbb{F}_{q^{1/2}}} a(-z)\eta^\star(1+yzi)\right) h_{tn_{0,\pi^{-1}y}} =$$

$$= \hat{a}(0)\cdot h_t + (a*\eta^\star)(0)\cdot\eta^\star(i)\cdot h_1 + \eta^\star(i)\cdot \sum_{0\neq y\in\mathbb{F}_q^-}\eta^\star(y)\cdot(a*\eta^\star)(i^{-1}y^{-1})\cdot h_{tn_{0,\pi^{-1}y}} =$$

$$= \hat{a}(0)\cdot h_t + (a*\eta^\star)(0)\cdot\eta^\star(i)\cdot r_1(\phi_1)$$

for all functions $a : \mathbb{F}_{q^{1/2}} \to \mathcal{O}_C$ such that $a * \eta^*$ is constant modulo $\lambda^{-1}\mathcal{O}_C$, where $\eta^\star$ is identified with its restriction on $\mathbb{F}_{q^{1/2}}^\times$.

When E/F is ramified, it is spanned by linear combinations

$$\sum_{z\in\mathbb{F}_q} a(-z)H_z = \left(\sum_{z\in\mathbb{F}_q} a(-z)\right)\cdot h_t + \left(\sum_{z\in\mathbb{F}_q} a(-z)\eta^{-1}(z)\right)\cdot h_1 + + \sum_{0\neq y\in\mathbb{F}_q}\left(\sum_{z\in\mathbb{F}_q} a(-z)\eta^{-1}(1+yz)\right) h_{tn_{0,\pi^{-1}y}} =$$

$$= \hat{a}(0)\cdot h_t + (a*\eta^{-1})(0)\cdot h_1 + \sum_{0\neq y\in\mathbb{F}_q}\eta^{-1}(y)\cdot(a*\eta^{-1})(y^{-1})\cdot h_{tn_{0,\pi^{-1}y}} =$$

$$= \hat{a}(0)\cdot h_t + (a*\eta^{-1})(0)\cdot\eta^{-1}(i)\cdot r_1(\phi_1)$$

for all functions $a : \mathbb{F}_q \to \mathcal{O}_C$ such that $a * \eta^{-1}$ is constant modulo $\lambda^{-1}\mathcal{O}_C$.

As $(a*\eta^\star)(0) \in \mathcal{O}_C$ and $\hat{a}(0) \in q^{1/2}\cdot\mathcal{O}_C + \lambda^{-1}\cdot\mathcal{O}_C$ by Lemma 7.17, we obtain when E/F is unramified that

$$r_1^{-1}(L_1^{'}) = \mathcal{O}_C\phi_1 + (q^{1/2}\lambda\cdot\mathcal{O}_C + \mathcal{O}_C)\cdot\phi_s$$

When E/F is ramified, as $(a*\eta^\star)(0) \in \mathcal{O}_C$ and $\hat{a}(0) \in q\cdot\mathcal{O}_C + \lambda^{-1}\cdot\mathcal{O}_C$ by Lemma 7.17, we obtain

$$r_1^{-1}(L_1^{'}) = \mathcal{O}_C\phi_1 + (q\lambda\cdot\mathcal{O}_C + \mathcal{O}_C)\cdot\phi_s$$

Note that when E/F is unramified, if $q^{1/2}\lambda \in \mathcal{O}_C$, i.e. $|\lambda| \le \left|q^{-1/2}\right|$, we are already done, as $r_1^{-1}(L_1^{'}) = r_0^{-1}(L_0) = \mathcal{O}_C\phi_1 + \mathcal{O}_C\phi_s$.

Similarly, when E/F is ramified, if $q\lambda \in \mathcal{O}_C$, i.e. $|\lambda| \le |q^{-1}|$, we are also done, as $r_1^{-1}(L_1^{'}) = r_0^{-1}(L_0) = \mathcal{O}_C\phi_1 + \mathcal{O}_C\phi_s$.

Assume, if so, in the case E/F unramified, that $q^{1/2}\lambda \notin \mathcal{O}_C$, so that $r_1^{-1}(L_1^{'}) = \mathcal{O}_C\phi_1 + q^{1/2}\lambda\mathcal{O}_C\phi_s$, and in the case E/F ramified, that $q\lambda \notin \mathcal{O}_C$, so that $r_1^{-1}(L_1^{'}) = \mathcal{O}_C\phi_1 + q\lambda\mathcal{O}_C\phi_s$. Then if E/F is unramified,

$$z(L_0) = K_0 \cdot \left(\mathcal{O}_C f_1 + q^{1/2}\lambda\mathcal{O}_C \cdot \sum_{n\in\mathbf{N}(k_F)} f_{sn}\right) = L_0 + q^{1/2}\lambda M_0$$

is the $\mathcal{O}_C$-module spanned over $\mathcal{O}_C$ by (see Proposition 7.11)

$$L_0, q^{1/2}\lambda \cdot \sum_{n\in\mathbf{N}(k_F)} f_{sn}, \left(q^{1/2}\lambda \cdot F_n\right)_{n\in\mathbf{N}(k_F)}$$

while if E/F is ramified,

$$z(L_0) = K_0 \cdot \left(\mathcal{O}_C f_1 + q\lambda\mathcal{O}_C \cdot \sum_{n\in\mathbf{N}'(k_F)} f_{sn}\right) = L_0 + q\lambda M_0$$

is the $\mathcal{O}_C$-module spanned over $\mathcal{O}_C$ by (see Proposition 7.11)

$$L_0, q\lambda \cdot \sum_{n\in\mathbf{N}'(k_F)} f_{sn}, (q\lambda \cdot F_n)_{n\in\mathbf{N}'(k_F)}$$

The $\mathcal{O}_C$-module $r_0^{-1}(z(L_0)) = (z(L_0))^{I(1)}$ is spanned, when E/F is unramified, over $\mathcal{O}_C$ by $L_0^{I(1)}$, $q^{1/2}\lambda\phi_s$ and by the preimages of

$$\begin{aligned}\sum_{n_{b,z}\in\mathbf{N}(k_F)} a(-b,\overline{z}) \cdot F_{b,z} \quad = \quad & \left(\sum_{n_{b,z}\in\mathbf{N}(k_F)} a(-b,\overline{z})\right) \cdot f_1 + \\ & + \sum_{n_{c,y}\in\mathbf{N}(k_F)} \left(\sum_{n_{b,z}\in\mathbf{N}(k_F)} a(-b,\overline{z}) \cdot \eta^{\star}(y+z-\overline{b}c)\right) f_{sn_{c,y}}\end{aligned}$$

for all functions $a : \mathbb{F}_q^N \to q^{1/2}\lambda\mathcal{O}_C$ such that $(c,y) \mapsto \sum_{n_{b,z}\in\mathbf{N}(k_F)} a(-b,\overline{z}) \cdot \eta^{\star}(y+z-\overline{b}c)$ is constant modulo $\mathcal{O}_C$.

When E/F is ramified, and $p \neq 2$, it is spanned over $\mathcal{O}_C$ by $L_0^{I(1)}$, $q\lambda\phi_s$ and by the preimages of

$$\sum_{b\in\mathbb{F}_q} a(-b) \cdot F_b = \left(\sum_{b\in\mathbb{F}_q} a(-b)\right) \cdot f_1 + \sum_{c\in\mathbb{F}_q} \left(\sum_{b\in\mathbb{F}_q} a(-b) \cdot \eta^{-1}\left(-\frac{(b+c)^2}{2}\right)\right) f_{sn_c}$$

for all functions $a : \mathbb{F}_q \to q\lambda\mathcal{O}_C$ such that $c \mapsto \sum_{b\in\mathbb{F}_q} a(-b) \cdot \eta^{-1}\left(-\frac{(b+c)^2}{2}\right)$ is constant modulo $\mathcal{O}_C$.

When E/F is ramified and $p = 2$, it is spanned over $\mathcal{O}_C$ by $L_0^{I(1)}$, $q\lambda\phi_s$ and by the preimages of

$$\sum_{z\in\mathbb{F}_q} a(-z)\cdot F_z = \left(\sum_{z\in\mathbb{F}_q} a(-z)\right)\cdot f_1 + \sum_{y\in\mathbb{F}_q}\left(\sum_{z\in\mathbb{F}_q} a(-z)\cdot\eta^{-1}(y+z)\right) f_{sn_y}$$

for all functions $a : \mathbb{F}_q \to q\lambda\mathcal{O}_C$ such that $y \mapsto \sum_{z\in\mathbb{F}_q} a(-z)\cdot\eta^{-1}(y+z)$ is constant modulo $\mathcal{O}_C$.

Definition 7.18. Let us now define for any two functions $a, \theta : N(k_F) \to C$ the *convolution of* a *with* θ*, which is the* function $a * \theta : \mathbf{N}(k_F) \to C$ defined by

$$(a*\theta)(x) = \sum_{n\in\mathbf{N}(k_F)} a(n^{-1})\cdot\theta(xn)$$

Or more explicitly

$$(a*\eta)(n_{c,y}) := \sum_{n_{b,z}\in\mathbf{N}(k_F)} a(n_{-b,\overline{z}})\cdot\theta(n_{b+c,y+z-\overline{b}c})$$

This convolution operation is associative, since for any $x \in \mathbf{N}(k_F)$ we have

$$((a*\theta)*\vartheta)(x) = \sum_{n\in\mathbf{N}(k_F)} (a*\theta)(n^{-1})\cdot\vartheta(xn) = \sum_{n\in\mathbf{N}(k_F)}\sum_{y\in\mathbf{N}(k_F)} a(y^{-1})\cdot\theta(n^{-1}y^{-1})\cdot\vartheta(xn) =$$

$$= \sum_{y\in\mathbf{N}(k_F)} a(y^{-1})\cdot\sum_{n\in\mathbf{N}(k_F)}\theta(n^{-1})\cdot\vartheta(xyn) = \sum_{y\in\mathbf{N}(k_F)} a(y^{-1})\cdot(\theta*\vartheta)(xy) = (a*(\theta*\vartheta))(x)$$

It is also clearly C-bilinear.

Similarly, we define a convolution on $\mathbf{N}'(k_F)$ in the same manner.

Definition 7.19. For a character $\eta : \mathbb{F}_q^\times \to \mathcal{O}_C^\times$, we extend it to a function $\eta : \mathbb{F}_q \to \mathcal{O}_C$ by setting $\eta(0) := 0$, and define a function $\tilde{\eta} : \mathbf{N}(k_F) \to C$ by setting for any $n_{b,z} \in \mathbf{N}(k_F)$

$$\tilde{\eta}(n_{b,z}) = \eta(z)$$

If E/F is ramified, we define $\tilde{\eta}(n_b) = -\frac{b^2}{2}$ if $p \neq 2$, and $\tilde{\eta}(n_{0,z}) = \eta(z)$ if $p = 2$.

Before we prove a similar proposition for this case, we prove a little lemma.

Lemma 7.20. *Assume $\eta \neq 1$. Then for any $1 \neq n \in \mathbf{N}(k_F)$, we have, if E/F is unramified,*

$$(\tilde{\eta}*\tilde{\eta^\star})(n) = \begin{cases} -1 & \eta\restriction_{\mathbb{F}_{q^{1/2}}} \neq 1 \\ -q^{1/2}(q^{1/2}-1)\eta^\star(i)\cdot\tilde{\eta}(n) - 1 & \eta\restriction_{\mathbb{F}_{q^{1/2}}} = 1, \quad p \neq 2 \\ -q^{1/2}(q^{1/2}-1)\tilde{\eta}(n) - 1 & \eta\restriction_{\mathbb{F}_{q^{1/2}}} = 1, \quad p = 2 \end{cases}$$

and if E/F is ramified, and $p \neq 2$,

$$(\tilde{\eta} * \tilde{\eta^{-1}})(n) = \begin{cases} -1 & \eta \neq \varepsilon_q, 1, n \neq 1 \\ q-2 & \eta = \varepsilon_q, 1, n \neq 1 \\ q-1 & n = 1 \end{cases}$$

Proof. Before we proceed, we note that for any $z \in \mathbb{F}_q$, as the norm map $\mathbb{F}_q \to \mathbb{F}_{q^{1/2}}$ is surjective, there exists $b \in \mathbb{F}_q$ such that $b\overline{b} = -(z + \overline{z})$.
Moreover, if $z \in \mathbb{F}_q^-$, it follows that $b = 0$ is the only possible value for b, and else, we have $\left|\mathbb{F}_q^1\right| = q^{1/2} + 1$ different solutions for this equation.
If $(c, y) \neq (0, 0)$, in particular $y \neq 0$, so we may consider the set $A = \left\{ \frac{z}{\overline{y}} \mid z \in \mathbb{F}_q^- \right\} = \frac{1}{\overline{y}} \cdot \mathbb{F}_q^-$.
For any $a \in \mathbb{F}_q$, if $1 \neq a \notin A$, then $a \cdot \overline{y} \notin \mathbb{F}_q^-$, hence $a \cdot \overline{y} + \overline{a} \cdot y \neq 0$, so there exist $q + 1$ different values of $d \in \mathbb{F}_q$ such that $d\overline{d} = -a \cdot \overline{y} - \overline{a} \cdot y$.
For each such value, we let $b = \frac{c+d}{a-1}$, which is well defined since $a \neq 1$, and $z = \frac{y - \overline{b}c}{a-1}$. Then

$$1 + \frac{y - \overline{b}c}{z} = a$$

$$z+\overline{z}+b\overline{b} = \frac{1}{(a-1)(\overline{a}-1)} \cdot \left(y(\overline{a}-1) - c(\overline{c}+\overline{d}) + \overline{y}(a-1) - \overline{c}(c+d) + (c+d)(\overline{c}+\overline{d})\right) =$$

$$= \frac{\overline{a}y + a\overline{y} - y - \overline{y} - 2c\overline{c} - c\overline{d} - \overline{c}d + c\overline{c} + c\overline{d} + \overline{c}d + d\overline{d}}{(a-1)(\overline{a}-1)} =$$

$$= \frac{(\overline{a}y + a\overline{y} + d\overline{d}) - (y + \overline{y} + c\overline{c})}{(a-1)(\overline{a}-1)} = 0$$

If $1 \neq a \in A$, then we see that there exists a unique such value of d, and b, z are chosen once more in the same way.
For $a = 1$, we seek solutions for $y - \overline{b}c = 0$. In such a case, we have either $c = 0$ (equivalently, $y \in \mathbb{F}_q^-$ or $1 \in A$), when we do not have any solutions, or $c \neq 0$ (equivalently, $y \notin \mathbb{F}_q^-$ or $1 \notin A$), when $b = \frac{\overline{y}}{\overline{c}}$ yields $q^{1/2}$ solutions. This shows that when $c = 0$

$$\sum_{1 \neq n_{b,z} \in \mathbf{N}(k_F)} \eta^\star \left(1 + \frac{y - \overline{b}c}{z}\right) = \sum_{1 \neq a \in A} \eta^\star(a) + (q^{1/2}+1) \cdot \sum_{a \notin A} \eta^\star(a) =$$

$$= \left(\sum_{a \in A} \eta^\star(a) + (q^{1/2}+1) \cdot \sum_{a \notin A} \eta^\star(a) \right) - 1$$

and when $c \neq 0$

$$\sum_{1 \neq n_{b,z} \in \mathbf{N}(k_F)} \eta^\star \left(1 + \frac{y - \overline{b}c}{z}\right) = \sum_{a \in A} \eta^\star(a) + q^{1/2} \cdot \eta^\star(1) + (q^{1/2}+1) \cdot \sum_{1 \neq a \notin A} \eta^\star(a) =$$

$$= \left(\sum_{a \in A} \eta^\star(a) + (q^{1/2}+1) \cdot \sum_{a \notin A} \eta^\star(a) \right) - 1$$

We note that $\sum_{a\in A} \eta^\star(a) = \eta(y) \cdot \sum_{a \in \mathbb{F}_q^-} \eta^\star(a)$, so in any case we are interested in computing the value of

$$\eta(y) \cdot \left(\sum_{a \in \mathbb{F}_q^-} \eta^\star(a) + (q^{1/2}+1) \cdot \sum_{a \notin \mathbb{F}_q^-} \eta^\star(a) \right) - 1$$

Now, we have already seen in Lemma2.24, that if $\eta \restriction_{\mathbb{F}_{q^{1/2}}} = 1$, $\sum_{a \in \mathbb{F}_q^-} \eta^\star(a) = q^{1/2} \cdot \eta^\star(i)$ if $p \neq 2$ and q if $p = 2$, while if $\eta \restriction_{\mathbb{F}_{q^{1/2}}} \neq 1$, $\sum_{a \in \mathbb{F}_q^-} \eta^\star(a) = 0$.

Note also that $\sum_{a \notin \mathbb{F}_q^-} \eta^\star(a) = -\sum_{a \in \mathbb{F}_q^-} \eta^\star(a)$, since $\eta \neq 1$ is nontrivial. Therefore, if $\eta \restriction_{\mathbb{F}_{q^{1/2}}} = 1$, then

$$(\tilde{\eta} * \tilde{\eta^\star})(n_{c,y}) = \eta(y) \cdot \left(-q \cdot \sum_{a \in \mathbb{F}_q^-} \eta^\star(a) \right) - 1 = \begin{cases} -q^{1/2}(q^{1/2}-1)\eta^\star(i) \cdot \eta(y) - 1 & p \neq 2 \\ -q^{1/2}(q^{1/2}-1)\eta(y) - 1 & p = 2 \end{cases}$$

Next, consider the case E/F is ramified and $p \neq 2$. Then, as for any $c \neq 0$, the map $b \mapsto 1 + \frac{c}{b}$ is bijective $\mathbb{F}_q^\times \to \mathbb{F}_q \backslash \{1\}$, we see that for $c \neq 0$

$$(\tilde{\eta} * \tilde{\eta^{-1}})(n_c) = \sum_{0 \neq b \in \mathbb{F}_q} \eta^{-1} \left(\left(1 + \frac{c}{b}\right)^2 \right) = \sum_{1 \neq x \in \mathbb{F}_q} \eta^{-1} \left(x^2\right)$$

and by Lemma 2.24, it follows that

$$(\tilde{\eta} * \tilde{\eta^{-1}})(n_c) = \begin{cases} -1 & \eta \neq \varepsilon_q, 1 \\ q-2 & \eta = \varepsilon_q, 1 \end{cases}$$

When $c = 0$, we see that $(\tilde{\eta} * \tilde{\eta^{-1}})(1) = \sum_{b \in \mathbb{F}_q^\times} \eta^{-1}(1) = q - 1$.

This concludes the proof of the lemma. ☐

We shall prove the following proposition.

Proposition 7.21. *When E/F is unramified, $\sum_{n \in \mathbf{N}(k_F)} a(n) \in \mathcal{O}_C$ if $a * \tilde{\eta}$ is constant modulo $\mathcal{O}_C$. When E/F is ramified, $\sum_{n \in \mathbf{N}'(k_F)} a(n) \in \mathcal{O}_C$ if $a * \tilde{\eta}$ is constant modulo $\mathcal{O}_C$.*

Proof. When E/F is ramified and $p = 2$, this is the result of Lemma 7.17. From now on, when E/F is ramified, we assume $p \neq 2$.

When the character η is trivial, the function $a * \tilde{\eta} + a = \sum_{n \in \mathbf{N}(k_F)} a(n)$ is constant. If $a * \tilde{\eta}$ is constant modulo $\mathcal{O}_C$, so is a, and

$$\sum_{n \in \mathbf{N}(k_F)} a(n) \in q^{3/2} a(1) + \mathcal{O}_C \subset q^2 \lambda \mathcal{O}_C + \mathcal{O}_C \subset \mathcal{O}_C$$

Similarly, when E/F is ramified, and $\eta = \varepsilon_q$ or trivial, the function $a * \tilde{\eta} + a = \sum_{n \in \mathbf{N}'(k_F)} a(n)$ is constant. If $a * \tilde{\eta}$ is constant modulo $\mathcal{O}_C$, so is a, and

$$\sum_{n \in \mathbf{N}'(k_F)} a(n) \in qa(1) + \mathcal{O}_C \subset q^2 \lambda \mathcal{O}_C + \mathcal{O}_C \subset \mathcal{O}_C$$

This settles the case of χ unramified. We turn to the tamely ramified case. If E/F is ramified, we may further assume $\mu \neq \varepsilon_q$.

We denote by $\mathcal{R}$ the space of integral functions $\mathbf{N}(k_F) \to \mathcal{O}_C$, by $\delta_0 \in \mathcal{R}$ the characteristic function of 0, and by $\Delta \in \mathcal{R}$ the constant function $\Delta = 1$.

Then, when E/F is unramified, we have $a \in q^{1/2} \lambda \mathcal{R}$ such that $a * \tilde{\eta} \in C \cdot \Delta + \mathcal{R}$, hence there exist $\alpha \in C$ and $\phi \in \mathcal{R}$ such that

$$a * \tilde{\eta} = \alpha \Delta + \phi \tag{7.7}$$

When E/F is ramified, we have that $a \in q\lambda\mathcal{R}$. Let us convolve by $\tilde{\eta^\star}$ when E/F is unramified, or by $\widetilde{\eta^{-1}}$ when E/F is ramified. Note that we have the following identities

$$(\tilde{\eta} * \tilde{\eta^\star})(n_{c,y}) = \sum_{n_{b,z} \in \mathbf{N}(k_F)} \eta(\bar{z}) \cdot \eta^\star(y + z - \bar{b}c) =$$

$$= \sum_{1 \neq n_{b,z} \in \mathbf{N}(k_F)} \eta^\star \left(\frac{y + z - \bar{b}c}{z} \right) = \sum_{1 \neq n_{b,z} \in \mathbf{N}(k_F)} \eta^\star \left(1 + \frac{y - \bar{b}c}{z} \right)$$

$$(\tilde{\eta} * \widetilde{\eta^{-1}})(n_c) = \sum_{b \in \mathbb{F}_q} \eta \left(-\frac{b^2}{2} \right) \cdot \eta^{-1} \left(-\frac{(b+c)^2}{2} \right) = \sum_{0 \neq b \in \mathbb{F}_q} \eta^{-1} \left(\left(1 + \frac{c}{b} \right)^2 \right)$$

It follows that when E/F is unramified, $(\tilde{\eta} * \tilde{\eta^\star})(1) = |\mathbf{N}(k_F)| - 1 = q^{3/2} - 1$, and when E/F is ramified, $(\tilde{\eta} * \widetilde{\eta^{-1}})(1) = q - 1$ (see Lemma 2.23).

Now, by Lemma 7.20 for $\eta : \mathbb{F}_q^\times \to C$, one has when E/F is unramified

$$\tilde{\eta} * \tilde{\eta^\star} = \begin{cases} q^{3/2} \cdot \delta_0 - \Delta & \eta \restriction_{\mathbb{F}_{q^{1/2}}} \neq 1 \\ q^{3/2} \cdot \delta_0 - \Delta - q^{1/2}(q^{1/2} - 1)\eta^\star(i) \cdot \tilde{\eta} & \eta \restriction_{\mathbb{F}_{q^{1/2}}} = 1, \quad p \neq 2 \\ q^{3/2} \cdot \delta_0 - \Delta - q^{1/2}(q^{1/2} - 1) \cdot \tilde{\eta} & \eta \restriction_{\mathbb{F}_{q^{1/2}}} = 1, \quad p = 2 \end{cases}$$

and when E/F is ramified

$$\tilde{\eta} * \widetilde{\eta^{-1}} = q \cdot \delta_0 - \Delta$$

Therefore, when E/F is unramified, if $\eta \restriction_{\mathbb{F}_{q^{1/2}}} \neq 1$, from (7.7) we have

$$q^{3/2} \cdot (a * \delta_0) - (a * \Delta) = a * (q^{3/2} \cdot \delta_0 - \Delta) = a * (\tilde{\eta} * \tilde{\eta^\star}) =$$

$$= (a * \tilde{\eta}) * \tilde{\eta^\star} = (\alpha \cdot \Delta + \phi) * \tilde{\eta^\star} = \alpha \cdot \left(\Delta * \tilde{\eta^\star} \right) + \phi * \tilde{\eta^\star}$$

and if E/F is ramified, we have

$$q \cdot (a * \delta_0) - (a * \Delta) = a * (q \cdot \delta_0 - \Delta) = a * (\tilde{\eta} * \tilde{\eta^{-1}}) =$$

$$= (a * \tilde{\eta}) * \tilde{\eta^{-1}} = (\alpha \cdot \Delta + \phi) * \tilde{\eta^{\star}} = \alpha \cdot \left(\Delta * \tilde{\eta^{-1}} \right) + \phi * \tilde{\eta^{-1}}$$

However, note that for any function a, one has when E/F is unramified

$$(a * \delta_0)(n_{c,y}) = \sum_{n_{b,z} \in \mathbf{N}(k_F)} a(n_{-b,\bar{z}}) \cdot \delta_0(n_{b+c,y+z-\bar{b}c}) = a(n_{c,y}) \Rightarrow a * \delta_0 = a$$

and when E/F is ramified

$$(a * \delta_0)(n_c) = \sum_{b \in \mathbb{F}_q} a(n_{-b}) \cdot \delta_0(n_{b+c}) = a(n_c) \Rightarrow a * \delta_0 = a$$

Furthermore, when E/F is unramified, we have

$$(a * \Delta)(n_{c,y}) = \sum_{n \in \mathbf{N}(k_F)} a(n) = (\Delta * a)(n_{c,y}) \Rightarrow a * \Delta = \Delta * a = \left(\sum_{n \in \mathbf{N}(k_F)} a(n) \right) \cdot \Delta$$

and when E/F is ramified

$$(a * \Delta)(n_y) = \sum_{b \in \mathbb{F}_q} a(n_b) = (\Delta * a)(n_c) \Rightarrow a * \Delta = \Delta * a = \left(\sum_{b \in \mathbb{F}_q} a(n_b) \right) \cdot \Delta$$

In particular, if $a = \tilde{\eta}$ is inflated from some character, and E/F is unramified, then

$$\sum_{n_{b,z} \in \mathbf{N}(k_F)} \eta(z) = \sum_{z \in \mathbb{F}_q^-} \eta(z) + (q+1) \cdot \sum_{z \notin \mathbb{F}_q^-} \eta(z) =$$

$$= \begin{cases} 0 & \eta \restriction_{\mathbb{F}_{q^{1/2}}} \neq 1 \\ -q^{1/2}(q^{1/2}-1)\eta(i) & \eta \restriction_{\mathbb{F}_{q^{1/2}}} = 1, \quad p \neq 2 \\ -q^{1/2}(q^{1/2}-1) & \eta \restriction_{\mathbb{F}_{q^{1/2}}} = 1, \quad p = 2 \end{cases}$$

if $a = \tilde{\eta}$, but E/F is ramified, then

$$\sum_{b \in \mathbb{F}_q} \eta \left(-\frac{b^2}{2} \right) = 0$$

Therefore, subtituiting, we obtain, when $\eta \restriction_{\mathbb{F}_{q^{1/2}}} \neq 1$, that if E/F is unramified

$$q^{3/2} a - \left(\sum_{n \in \mathbf{N}(k_F)} a(n) \right) \cdot \Delta = \phi * \tilde{\eta^{\star}} \in \mathcal{R}$$

and if E/F is ramified

$$qa - \left(\sum_{b \in \mathbb{F}_q} a(n_b) \right) \cdot \Delta = \phi * \tilde{\eta^{-1}} \in \mathcal{R}$$

However, as $a \in q^{1/2}\lambda\mathcal{R}$ when E/F is unramified, we see that $q^{3/2}a \in q^2\lambda\mathcal{R} \subset \mathcal{R}$, hence $\sum_{n \in \mathbf{N}(k_F)} a(n) \in \mathcal{O}_C$, as required.

Also, as $a \in q\lambda\mathcal{R}$ when E/F is ramified, we see that $qa \in q^2\lambda\mathcal{R} \subset \mathcal{R}$, hence $\sum_{b \in \mathbb{F}_q} a(n_b) \in \mathcal{O}_C$, as required.

Similarly, if $\eta \restriction_{\mathbb{F}_{q^{1/2}}} = 1$, we obtain from (7.7) either

$$q^{3/2}a - \left(\sum_{n \in \mathbf{N}(k_F)} a(n) \right) \cdot \Delta - q^{1/2}(q^{1/2} - 1) \cdot (a * \tilde{\eta}) =$$

$$= a * (q^{3/2} \cdot \delta_0 - \Delta - q^{1/2}(q^{1/2} - 1) \cdot \tilde{\eta}) =$$
$$= \alpha \cdot \left(\Delta * \tilde{\eta^\star}\right) + \phi * \tilde{\eta^\star} = -q^{1/2}(q^{1/2} - 1)\alpha \cdot \Delta + \phi * \tilde{\eta^\star}$$

in the case $p = 2$, which after resubtituting, becomes

$$q^{3/2}a - \left(\sum_{n \in \mathbf{N}(k_F)} a(n) \right) \cdot \Delta - q^{1/2}(q^{1/2} - 1)\phi = \phi * \tilde{\eta^\star}$$

or

$$q^{3/2}a - \left(\sum_{n \in \mathbf{N}(k_F)} a(n) \right) \cdot \Delta - q^{1/2}(q^{1/2} - 1)\eta^\star(i) \cdot (a * \tilde{\eta}) =$$
$$= -q^{1/2}(q^{1/2} - 1)\eta^\star(i) \cdot \alpha \cdot \Delta + \phi * \tilde{\eta^\star}$$

in the case $p \neq 2$, which after resubtituting, becomes

$$q^{3/2}a - \left(\sum_{n \in \mathbf{N}(k_F)} a(n) \right) \cdot \Delta - q^{1/2}(q^{1/2} - 1)\eta^\star(i) \cdot \phi = \phi * \tilde{\eta^\star}$$

In any case, by the same argument, since $q^{1/2}(q^{1/2} - 1), \eta^\star(i) \in \mathcal{O}_C$, we see that $\sum_{n \in \mathbf{N}(k_F)} a(n) \in \mathcal{O}_C$, as required. □

Corollary 7.22. *If $\lambda^{-1} \in \mathcal{O}_C$, $q^2\lambda \in \mathcal{O}_C$, then $ind_B^G \chi$ is integral.*

Proof. Returning to the $\mathcal{O}_C$-module $r_0^{-1}(z(L_0)) = (z(L_0))^{I(1)}$, we see that it is generated by $\phi_1, q^{1/2}\lambda\phi_s$ and by

$$\left(\sum_{n \in \mathbf{N}(k_F)} a(n^{-1}) \right) \cdot \phi_1 + (a * \tilde{\eta^\star})(0) \cdot \phi_s$$

for all functions $a : \mathbf{N}(k_F) \to q^{1/2}\lambda\mathcal{O}_C$ such that $a * \tilde{\eta^\star}$ is constant modulo $\mathcal{O}_C$. For such functions, by Proposition 7.21, we have $\sum_{n\in\mathbf{N}(k_F)} a(n) \in \mathcal{O}_C$, hence the above functions already belong to $\mathcal{O}_C\phi_1 + q^{1/2}\lambda\mathcal{O}_C\phi_s$, showing that $r_0^{-1}(z(L_0)) = \mathcal{O}_C\phi_1 + q^{1/2}\lambda\mathcal{O}_C\phi_s = r^{-1}(L_1)$, hence the coefficient system corresponding to $z(L_0), L_1$ yields an integral structure in $ind_B^G\chi$, by Corollary 6.16. □

Remark 7.23. (i) In Theorem 7.1, $\chi(\pi)$ is a unit if and only if the character $\chi \otimes \chi_1$ of M is $\mathcal{O}_C$-integral. By Corollary 7.8, in this case, L is the natural $\mathcal{O}_C$-integral structure of functions in $ind_B^G(\chi \otimes \chi_1)$ with values in $\mathcal{O}_C$. The reduction of L is the k_C-principal series of G induced from the reduction $\overline{\chi}\otimes\overline{\chi}_1$ of $\chi \otimes \chi_1$.

(ii) The unimodular character of P is $\delta(p) = \omega(a)^2$, where $\omega(\pi) = 1/q^2$. The contragredient of $ind_B^G(\chi\otimes\chi_1)$ is $ind_B^G(\chi^{-1}\cdot\omega^2\otimes\chi_1^{-1})$, hence Theorem 7.1 and Corollary 6.21 are compatible.

Let $\chi_0(t) = \chi(t)\chi_1(t\bar{t}^{-1})$. Then D. Keys showed in Keys [18] that the representation

$$ind_B^G(\chi \otimes \chi_1) \simeq ind_B^G(\chi^\star\omega^2 \otimes \chi_1)$$

is reducible only when either $\chi_0 = 1, \omega^2$, or $\chi_0 \in \left\{\eta\omega^{1/2}, \eta\omega^{3/2}\right\}$ and $\eta \restriction_{F^\times} = \eta_{E/F}$, or $\chi_0 \restriction_{F^\times} = \omega$ but $\chi_0 \neq \omega$. The isomorphism is compatible with Theorem 7.1

(iii) Theoretically, there is no reason to restrict to the tamely ramified smooth case, but the computations become harder when the level increases or when one adds an algebraic part.

(iv) One should see c) as the limit at ∞ of the integrality local criterion.

8. Application to reduction and k-representations

8.1. Reduction.

An R-integral finitely generated S-representation V of G contains an R-integral structure L_{ft} which is finitely generated as an RG-module; two finitely generated R-integral structures L_{ft}, L'_{ft} of V are commensurable: there exists $a \in R$ non zero such that $aL_{ft} \subset L'_{ft}$, $aL'_{ft} \subseteq L_{ft}$.

Let x be a uniformizer of R and $k = {}^{R}/_{xR}$. When the reduction $\overline{L}_{ft} := L_{ft}/xL_{ft}$ is a finite length kG-module, the reduction $\overline{L}$ of an R-integral structure L of V commensurable to L_{ft} has finite length and the same semi-simplification as $\overline{L}_{ft}$.

Lemma 8.1. *If the reduction $\overline{L}_{ft}$ is an irreducible k-representation of H, then the R-integral structures of V are the multiples of L_{ft}.*

Proof. Let L be an integral structure of V which is different from L_{ft}. Taking a multiple of L_{ft}, we reduce to $L_{ft} \subset L$ and L_{ft} not contained in xL. The inclusions

$$xL_{ft} \subset (xL \cap L_{ft}) \subset L_{ft}$$

the right one being strict, and the irreducibility of L_{ft}/xL_{ft} imply $xL \cap L_{ft} = xL_{ft}$, equivalent to $L = L_{ft}$ because there exists no $v \in L$ and $v \notin L_{ft}$ such that $v \in x^{-1}L_{ft}$. $\square$

In the integrality criterion (Proposition 6.13), when the properties of 2) are true, the reduction of the R-integral structure $H_0(\mathcal{L})$ of the S-representation $H_0(\mathcal{V})$ of G is the 0-th homology of the G-equivariant coefficient system defined by the diagram

$$\overline{L}_0 \longleftarrow \overline{L_{01}} \longrightarrow \overline{L}_1$$

We have the exact sequence of SG-modules:

$$0 \to ind_I^G V_{01} \to ind_{K_1}^G V_1 \oplus ind_{K_0}^G V_0 \to H_0(\mathcal{V}) \to 0$$

of free RG-modules:

$$0 \to ind_I^G L_{01} \to ind_{K_1}^G L_1 \oplus ind_{K_0}^G L_0 \to H_0(\mathcal{L}) \to 0$$

of kG-modules:

$$0 \to ind_I^G \overline{L}_{01} \to ind_{K_1}^G \overline{L}_1 \oplus ind_{K_0}^G \overline{L}_0 \to H_0(\overline{\mathcal{L}}) \to 0$$

8.2. k-representations of G.

Let k be a finite field of characteristic p. Consider an irreducible principal series of G over k. It can be written as $ind_B^G(\overline{\chi} \otimes \overline{\chi_1})$ for some lifts χ, χ_1.

By Theorem 7.1, Remark 7.23 (i), and Lemma 8.1, it follows that each $\mathcal{O}_C$-integral structure of $ind_B^G(\chi \otimes \chi_1)$ is a multiple of L which is defined in Remark 7.23 (i).

Therefore, an irreducible principal series of G over k is the 0-th homology of a G-equivariant coefficient system.

Let $\mu \otimes \mu_1$ be a k-character of M; its restriction to $M(\mathcal{O}_E)$ is the inflation of a k-character $\eta \otimes \eta_1$ of $M(\mathbb{F}_q)$. As before, $\left(ind_{\mathbf{B}(\mathbb{F}_q)}^{\mathbf{G}(\mathbb{F}_q)}(\eta \otimes \eta_1)\right)^{\mathbf{N}(\mathbb{F}_q)} = C \cdot \phi_1 \oplus C \cdot \phi_s$ where ϕ_1, ϕ_s have support $\mathbf{B}(\mathbb{F}_q)$, $\mathbf{B}(\mathbb{F}_q)s\mathbf{N}(\mathbb{F}_q)$ and value 1 at id, s, respectively.

Proposition 8.2. *The principal series $ind_B^G(\mu \otimes \mu_1)$ is the 0-th homology of the G-equivariant coefficient system defined by the tamely ramified diagram*

$$\begin{array}{ccc} & & \big(ind_B^G(\mu\otimes\mu_1)\big)^{K_0(1)} \\ & \nearrow & \\ \big(ind_B^G(\mu\otimes\mu_1)\big)^{I(1)} & & \\ & \searrow & \\ & & \big(ind_B^G(\mu\otimes\mu_1)\big)^{K_1(1)} \end{array}$$

inflated from the inclusions

$$\Big(ind_{\mathbf{B}(\mathbb{F}_q)}^{\mathbf{G}(\mathbb{F}_q)}(\eta \otimes \eta_1)\Big)^{\mathbf{N}(\mathbb{F}_q)} \to ind_{\mathbf{B}(\mathbb{F}_q)}^{\mathbf{G}(\mathbb{F}_q)}(\eta \otimes \eta_1)$$

$$\Big(ind_{\mathbf{M}(\mathbb{F}_q)\mathbf{Z}(\mathbb{F}_q)}^{\mathbf{H}(\mathbb{F}_q)}(\eta \otimes \eta_1)\Big)^{\mathbf{Z}(\mathbb{F}_q)} \to ind_{\mathbf{M}(\mathbb{F}_q)\mathbf{Z}(\mathbb{F}_q)}^{\mathbf{H}(\mathbb{F}_q)}(\eta \otimes \eta_1)$$

Proof. Let $\mu \otimes \mu_1 : M \to k^\times$ be a continuous character. There exists a moderately ramified continuous character $\chi \otimes \chi_1 : M \to \mathcal{O}_C^\times$ lifting $\mu \otimes \mu_1$. Apply Theorem 7.1 and Remark 7.23 (i). $\square$

[1] *Ramla Abdellatif.* Autour des représentations modulo p des groupes réductifs p-adiques de rang 1. *PhD thesis, Université Paris Sud-Paris XI, 2011.*

[2] *Peter Abramenko and Gabriele Nebe. Lattice chain models for affine buildings of classical type.* Mathematische Annalen*, 322(3):537–562, 2002.*

[3] *Eran Assaf, David Kazhdan, and Ehud de Shalit. Kirillov models and the breuil-schneider conjecture for $GL_2(F)$.* arXiv preprint arXiv:1302.3060*, 2013.*

[4] *Laure Barthel, Ron Livne, et al. Irreducible modular representations of* GL_2 *of a local field.* Duke Mathematical Journal*, 75(2):261–292, 1994.*

[5] *Christophe Breuil. The emerging p-adic langlands programme. In* Proceedings of the international congress of mathematicians*, volume 2, pages 203–230, 2010.*

[6] *Christophe Breuil and Peter Schneider. First steps towards p-adic langlands functoriality.* Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2007(610):149–180, 2007.*

[7] *François Bruhat and Jacques Tits. Groupes réductifs sur un corps local.* Publications mathématiques de l'IHES, *41(1):5–251, 1972.*

[8] *Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paskunas, and Sug Woo Shin. Patching and the p-adic local langlands correspondence.* arXiv preprint arXiv:1310.0831*, 2013.*

[9] *Marco De Ieso. Existence de normes invariantes pour GL_2.* Journal of Number Theory*, 133(8):2729–2755, 2013.*

[10] *Matthew Emerton. Locally analytic vectors in representations of locally p-adic analytic groups.* arXiv preprint math/0405137*, 2004.*

[11] *Matthew Emerton et al. p-adic L-functions and unitary completions of representations of p-adic reductive groups.* Duke Mathematical Journal*, 130(2):353–392, 2005.*

[12] *William Fulton and Joe Harris.* Representation theory*, volume 129. Springer Science & Business Media, 1991.*

[13] *Paul B Garrett.* Buildings and classical groups*. CRC Press, 1997.*

[14] *Elmar Große-Klönne. On the universal module of p-adic spherical hecke algebras.* American Journal of Mathematics*, 136(3):599–652, 2014.*

[15] *Yongquan Hu. Normes invariantes et existence de filtrations admissibles.* Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2009 (634):107–141, 2009.*

[16] *Jens C Jantzen. First cohomology groups for classical lie algebras. In* Representation theory of finite groups and finite-dimensional algebras*, pages 289–315. Springer, 1991.*

[17] *Shin-ichi Kato. On eigenspaces of the hecke algebra with respect to a good maximal compact subgroup of a p-adic reductive group.* Mathematische Annalen*, 257(1):1–7, 1981.*

[18] *David Keys. Principal series representations of special unitary groups over local fields.* Compositio mathematica*, 51(1):115–130, 1984.*

[19] *Karol Koziol and Peng Xu. Hecke modules and supersingular representations of U (2, 1).* arXiv preprint arXiv:1204.1273*, 2012.*

[20] *Vytautas Paskunas. Coefficient systems and supersingular representations of $GL_2(F)$.* arXiv preprint math/0403240*, 2004.*

[21] *Peter Schneider and Ulrich Stuhler. Representation theory and sheaves on the bruhat-tits building.* Publications Mathématiques de l'IHÉS*, 85(1): 97–191, 1997.*

[22] *Peter Schneider and Jeremy Teitelbaum. Locally analytic distributions and p-adic representation theory, with applications to GL_2.* Journal of the American Mathematical Society*, 15(2):443–468, 2002.*

[23] *Peter Schneider, Jeremy Teitelbaum, and Dipendra Prasad. $U(g)$-finite locally analytic representations.* Representation Theory of the American Mathematical Society*, 5(5):111–128, 2001.*

[24] *Peter Schneider, Jeremy Teitelbaum, Das Uferlose, and das Leben Stürmende. Banach-hecke algebras and p-adic galois representations.* Doc. Math*, pages 631–684, 2006.*

[25] *Claus M Sorensen. A proof of the breuil-schneider conjecture in the indecomposable case.* Annals of Mathematics*, 177:1–16, 2013.*

[26] *Jacques Tits. Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque.* Journal für die reine und angewandte Mathematik*, 247:196–220, 1971.*

[27] *Jacques Tits. Reductive groups over local fields. In* Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part*, volume 1, pages 29–69, 1979.*

[28] *Marie-France Vignéras.* Representations modulaires des groupes reductifs p-adiques. Representations cuspidales de $GL(n)$*, volume 137. Springer, 1996.*

[29] *Marie-France Vignéras. A criterion for integral structures and coefficient systems on the tree of $PGL(2, F)$.* Pure and Applied Mathematics Quarterly*, 4(4), 2008.*

[illegible] GL(2) [illegible]
[illegible] U(3) ([illegible]) [illegible]

[illegible]
[illegible] G [illegible]

,2 [illegible] GL(2) [illegible]

,3 [illegible] GL(2) [illegible]

,4 [illegible] U(3) [illegible]

[illegible]

תקציר

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

הוגש לסנט האוניברסיטה העברית בירושלים

נובמבר 2016

עבודה זו נעשתה בהדרכתו

של פרופ' אהוד דה-שליט

קיום נורמות אינוריאנטיות בהצגות p-אדיות

של חבורות רדוקטיביות

חיבור לשם קבלת תואר דוקטור לפילוסופיה

מאת

ערן אסף

קיום נורמות אינוריאנטיות בהצגות p-אדיות

של חבורות רדוקטיביות

חיבור לשם קבלת תואר דוקטור לפילוסופיה

מאת

ערן אסף